Coloring a Fence

Dynamic Programming

Picket Fence

Input: Fence = Pickets P_1 \ldots P_n
  Requested colors C_1 \ldots C_n

Operation: Stroke (i, j, c) = colors pickets P_i \ldots P_j \text{ color } c

Output: min stroke seq coloring the fence.

\[\begin{array}{c}
  1 & 2 & 3 & 4 & 5 \\
  R & B & R & B & R \\
\end{array}\]

\[S(1, 5, R), S(2, 3, R), S(4, 4, R)\]
Subproblems

\[
\text{Naive} \quad \#(i,j) = \# \text{strokes } i \to j
\]

\[
\#(i,j,c) = \min \# \text{strokes given free background color } c.
\]

Recurrence

\[
\#(i,j,c) = \begin{cases} 
0 & \text{if } j < i \\
0 & \text{if } i = j \land C_i = C \\
\min & \text{over } k < j \\
\text{a) } & \min \ C(i,k-1,c) + C(k+1,j,c) \\
\text{b) } & C(i,j,c) + 1 & C_j \neq C
\end{cases}
\]
Claim $\#(i, j, c) \geq \text{Opt}$

Recurse generating a coloring of $\#(i, j, c)$ strokes

Claim $\#(i, j, c) \leq \text{Opt}$

\* induction on $t = j - 1$

$t \leq 1$ done

assume true let Suppose $i - i = t + 1$

let $S = \{S_0, \ldots, S_k\}$ be opt seq of strokes.

Case 1 $S$ uses background color $c$. say $P_k$

$\exists$ partition of $S$ into $S' \& S''$

$S'$ coloring of $P_i \ldots P_{k-1}$ with by $c$.

$S''$, "P_{k1} \ldots P_{j1}"

by induction $\#(i, k - 1, c) \leq 15'$

$\#(k, j, i, c) \leq 15''$

Thus $\#(i, j, c) \leq 15$
Case 2. BG color not used

WLOG $P_j$ only pointed once
2) first stroke points $P_j$

$$S_1 = \text{stroke}(i, j, C_j)$$

$$S_2 = S_k \text{ paint } \overline{P_i - P_k} \text{ using bg color } C_j$$

$$\#(i, j, C) \leq \#(i, j-1, C_j + 1) \leq (k-1)^+ \leq 15$$

Timings subprobs

$$(i, j, c) O(n^3 \cdot c) \quad c = \# \text{ of colors.}$$

Cost per problem $O(n)$

$$O(n^3 \cdot c)$$