Probability 101

15-750
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Random Variable: Samples $\to \mathbb{R}$

Prob Density Fcn

eg X_β exponential random variable

$$\text{Prob}[X_\beta = \mu] = \begin{cases} \beta e^{-\beta x} & \mu \geq 0 \\ 0 & \text{o.w.} \end{cases}$$

Cumulative Dist

$$F_{X_\beta}(y) = \text{Prob}[X_\beta \leq y]$$

$$F_\beta(y) = \int_0^y \beta e^{-\beta x}\, dx \overset{\text{Fund thm of cal}}{\equiv} e^{-\beta x}\Big]_0^y = 1 - e^{-\beta y}$$

Def Expected Value

$$Ex[X] = \int_{-\infty}^{\infty} x \Pr[X = y]\, dy$$

$$Ex[X_\beta] = \int_0^\infty y \beta e^{-\beta y} dy = 1/\beta$$

Intergration by parts

$$Pr[X > m+n \mid X \geq m] = \frac{e^{-\beta(n+m)}}{e^{-\beta m}} = e^{-\beta n}$$

Memoryless Property

$Select_k(x_1, \dots, x_n)$ = kth smallest from $\{x_1, \dots, x_n\}$

Order Statistics

Random Variables $X_1 \dots X_n$

$X_{(k)} \equiv Select_k(X_1, \dots, X_n)$

Note $X_{(1)} \leq \dots \leq X_{(n)}$

$Pr[X_{(1)} = u] \equiv f_1(u)$ $\quad f(u) = Pr[X_i = u]$ for any i

$Pr[0 \le X_i \le u] = F(u)$

$$f_1(u) = n(1 - F(u))^{n-1} f(u) = n e^{-(n-1)\beta u} \beta e^{-\beta u}$$

$$= n\beta e^{-n\beta u} \equiv Exp(n\beta)$$

$Ex[f_1] = \frac{1}{n\beta}$

$S_i \equiv X_{(i+1)} - X_{(i)}$ $\quad i \geq 1$

by memoryless prop $S_i \equiv Exp((n-i)\beta)$

$Ex[S_i] = \frac{1}{(n-i)\beta}$

$$Ex[X_{(n)}] = \sum_{i=0}^{n-1} Ex[S_i] = \frac{1}{\beta}\left[1 + \frac{1}{2} + \cdots \frac{1}{n-1}\right] \approx \frac{\lg n}{\beta}$$

Concentration for $X_{(n)}$

$$\Pr\left\{X_i \geq \frac{c \ln n}{\beta}\right\} = e^{-\frac{c \ln n}{\beta}\beta} = n^{-c}$$

$$\Pr\left[X_{(n)} \geq \frac{c \ln n}{\beta}\right] \leq n \cdot n^{-c} = \frac{1}{n^{c-1}}$$

$$\Pr\left[X_{(n)} \geq \frac{2 \ln n}{\beta}\right] \leq \frac{1}{n}$$

Generating Dist

$f: \mathbb{R} \to \mathbb{R}^+$ PDF $\int_{-\infty}^{\infty} f(x)\,dx = 1$ X_f random variable

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Def $f, g \in$ PDF's we say $f \leq g$ if

$\exists$ det process D s.t. $X_f = D(X_g)$

U is uniform $[0,1]$ is $u(x) = \begin{cases} 1 & \text{if } x \in [0,1] \\ 0 & \text{o.w.} \end{cases}$

U_2 is uniform $[0,2]$

Claim $U_2 \leq U$ $X_{U_2} = 2X_U$

Generate Exponential Dist from uniform.

Pdf $\equiv$ $f(x) = \beta e^{-\beta x}$ $\quad 0 < \beta$ & $x \geq 0$

$$F(x) = \int_0^x f(x)\,dx = 1 - e^{-\beta x}$$

$F: [0, \infty] \to [0, 1]$ $\quad$ 1-1 & onto

Thus $\mathcal{U} \leq f$

Random Var $\quad y = F(x) = 1 - e^{-\beta x}$

$$e^{-\beta y} = 1 - y$$

$$-\beta x = \ln(1-y)$$

$$x = -\frac{1}{\beta}\ln(1-y)$$

Note y uniform $[0,1]$ then $1-y$ " "

$$x = \frac{-\ln(\mathcal{U})}{\beta}$$

$\text{Exp} \leq \mathcal{U}$