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Linear Programming in Fixed Dim

Goal: 1) Intro LP
2) Random incremental 2D Alg

Def LP $\equiv \max_x c^T x$ subject $Ax \le d$

$A^{n \times m}$ in $\mathbb{R}^{n \times m}$ $\wedge$ $x^{m \times 1}$ $\wedge$ $c^{m \times 1}$ $\wedge$ $d^{n \times 1}$

$x \le y$ if $\forall i \; x_i \le y_i$

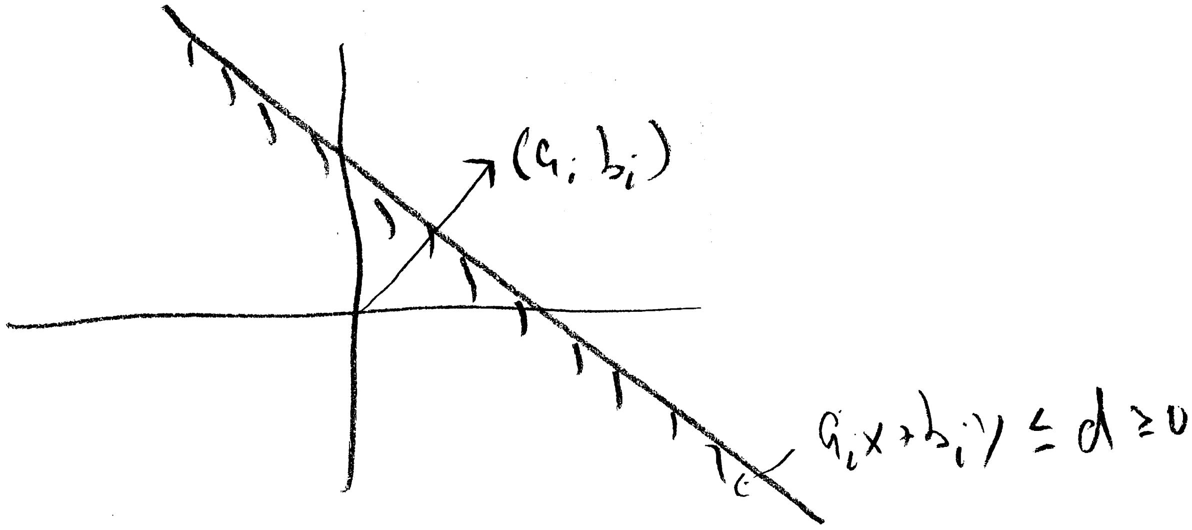
Def the LP is feasible $Ax \le d$

Note $\{x \mid Ax \le d\}$ is Convex

2D case

$$\begin{pmatrix} a_1 & b_1 \\ \vdots & \\ a_n & b_n \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \le \begin{pmatrix} d_1 \\ \vdots \\ d_n \end{pmatrix}$$

Note $h_i = \{(x,y) \mid a_i x + b_i y \le d_i\}$ Half-space
Half-plane



Sorting	Selection like probs
Half-Space-inter	LP
Convex Hull	?
Meshing	?

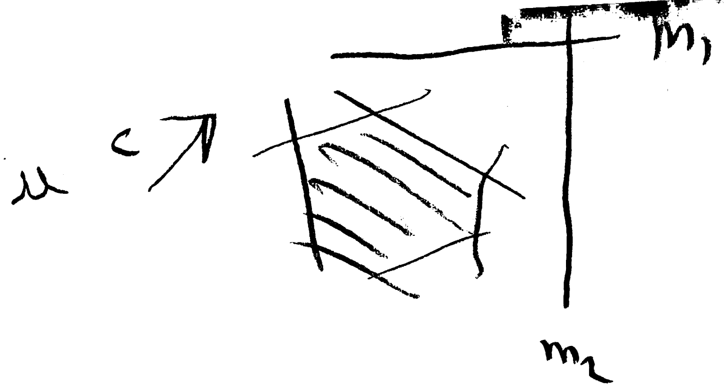
Geo 2D LP

<u>Input</u>: Half-Spaces $\{h_1, \ldots, h_n\}$ & vector c.

Simplify:

1) no h_i normal to c ie

2) Given bdling box for feasible region



1D-LP

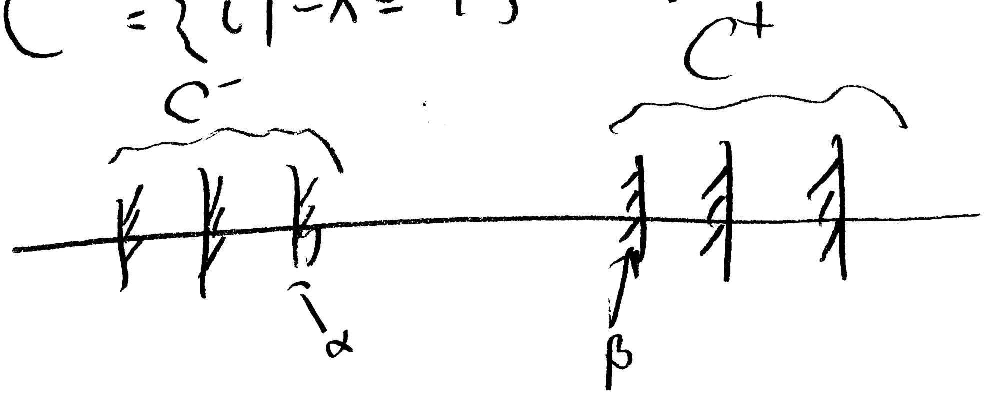
Input: Constraints $a_i x \le b_i$ $a_i \ne 0$

WLOG $a_i = \pm 1$

Consider

$C^+ = \{ i \mid x \le b_i \}$

$C^- = \{ i \mid -x \le b_i \}$ $-b_i \le x$



$\alpha = \max \{ -b_i \mid i \in C^- \}$

$\beta = \min \{ b_i \mid i \in C^+ \}$

feasible iff $\alpha \le \beta$

if feasible answer is $\begin{cases} \beta & \text{if } sign(c) = 1 \\ \alpha & \text{o.w.} \end{cases}$

<u>Thm</u> 1D-LP is $O(n)$ time

Return to 2D-LP prob

Procedure 2D-LP$(m_1, m_2, h_1 \cdots h_n, c)$

1) $V_0 \leftarrow$ 2D-LP(m_1, m_2, c)

   ie $V_0 = CH(m_1) \cap CH(m_2)$

2) Randomly order $h_1, \ldots, h_n$

3) <u>for</u> $i = 1$ to n <u>do</u>

   if $V_{i-1} \in h_i$ then $V_i \leftarrow V_{i-1}$

   else (Make 1D-LP prob) $L = CH(h_i)$

   $h_1' = L \cap h_1 \quad \cdots \quad h_{i-1}' = L \cap h_{i-1}$

   $c' = proj(c, L)$ note: $c' \neq 0$

   $V_i = $ 1D-LP$(h_1', \ldots, h_{i-1}', c')$

   if V_i is "undef" report "no-solution" & halt

Correctness:

Claim: At time (*) $V_i = LP(m_1, \ldots h_i)$

pf by induct

base OK

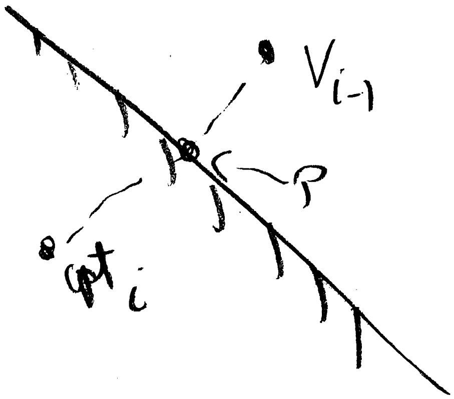
assume V_{i-1} is correct

case: $V_{i-1} \in h_i$ then $V_{i-1} \in$ feasible region $\Rightarrow$ opt

case: $V_{i-1} \notin h_i$

<u>Claim</u> $V_i \in CH(h_i) = L$

picture proof



$c^T V_{i-1} > c^T Opt_i$

$\Rightarrow c^T p > c^T Opt_i$

& p feasible

Timing

Claim 2D-LP is $O(n)$ expected time.

We use backward analysis

Suppose we remove h_j from $h_1 \cdots h_i$

Def h_j critical if removing it changes opt solution

eg

Note At most 2 critical constraints

$$\text{cost}(h_j) = \begin{cases} K & \text{if } h_j \text{ not critical} \\ Ki & \text{o.w} \end{cases}$$

Worst case exactly 2 critical

$$E_i = \left((2)\,ki + (i-2)\,k\right)/i = \frac{3ki}{i} = 3k$$

Total Expected cost $\sum_{i=1}^{n} 3k = O(n)$