Simplex Alg

15-750
2/15/10

understand the main idea (pivoting)

standard form max $C^T X$ subject $AX \leq b$ & $X \geq 0$

if A is $m \times n$ we have n variables

m+n constraints

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & & a_{mn} \end{pmatrix}$$

Def for some $\bar{X}$ constraint

$a_{i1}\bar{x}_1 \cdots a_{in}\bar{x}_n = b_i$ tight $\quad x_i = 0$

$a_{i1}\bar{x}_1 + \cdots + a_{in}\bar{x}_n < b_i$ slack $\quad x_i > 0$

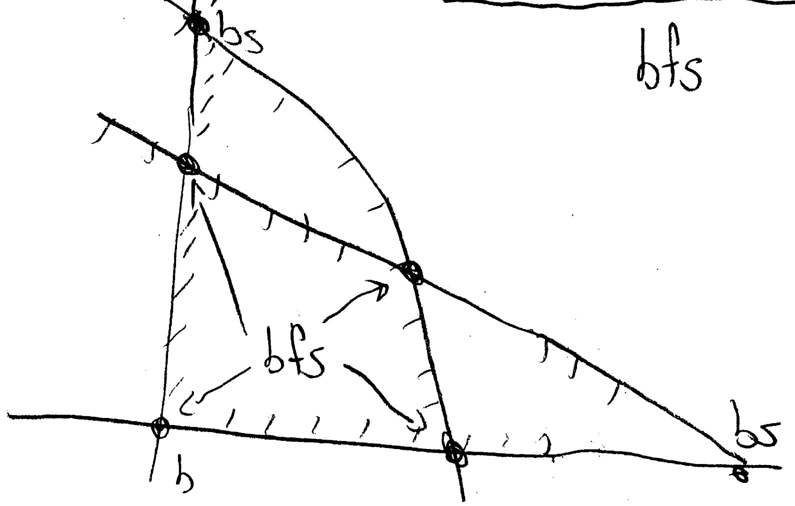
Def $\bar{X}$ is <u>basic</u> (bs) <u>solution</u> if n tight constraints

basic need <u>not</u> be feasible.

some constraints <u>not</u> satisfied.

Def $\bar{X}$ is also feasible we say $\bar{X}$ is a <u>basic feasible solution</u>

bfs



Slack Form

$\max c^T x$ subject $Ax \le b$ & $x \ge 0$

standard $\Rightarrow$ slack form

slack variable

eg $a_1 x_1 \cdots a_n x_n \le b \Rightarrow a_1 x_1 \cdots + a_n x_n + w = b \quad w \ge 0$

$$\begin{pmatrix} A & \begin{matrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{matrix} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \\ w_1 \\ \vdots \\ w_m \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} \qquad x \ge 0, w \ge 0$$

$$\begin{array}{cc} & m+n \\ m & \begin{pmatrix} A \end{pmatrix} \end{array} x = b \qquad x \ge 0$$

note: A constraint is tight iff $x_i = 0$.
x_i is either an old variable or a slack one.

To get a basic solution we set n variables to zero.

A basic solution determined by n variables set to zero

Let $B \subseteq \{1, \ldots, m+n\} \quad |B| = n$

After reording the columns of A and the variables

$$\begin{array}{c|c|c|} & m & n \\ m & A_B & A_{\bar{B}} \end{array} \begin{pmatrix} x_1 \\ \vdots \\ x_m \\ 0 \\ \vdots \\ 0 \end{pmatrix} = b \qquad \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} = x_B \qquad \begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix} = c_B$$

solve $A_B x_B = b$ $\quad \begin{pmatrix} x_B \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ is a bfs objective $c_B^T x_B$

Note B is feasible iff $x_B \geq 0$

Suppose B is bfs Goal: $B' = B - \{i\} + \{j\}$

B' bfs

$c_B^T x_B < c_{B'}^T x_{B'}$

Question: which i & j? Idea: pick i & compute j.

after Gaussian Elimination

new b. $\quad$ not degenerate

$$\begin{array}{c|c|c|} & m & n \\ m & I & A_{\bar{B}} \end{array} \begin{pmatrix} x_B \\ 0 \end{pmatrix} = b \qquad b = x_B > 0$$

relax constraint $i \in B$ ie $\begin{pmatrix} I & \cdots & A_i \end{pmatrix} \begin{pmatrix} x_B \\ 0 \\ \vdots \\ 0 \\ \varepsilon \\ 0 \\ \vdots \\ 0 \end{pmatrix} = b$

$$\begin{pmatrix} I & A_i \end{pmatrix} \begin{pmatrix} x_B \\ \varepsilon \end{pmatrix} = b$$

$$x_B = (b - \varepsilon A_i) > 0$$

$\therefore$ small ε still feasible

pick first j s.t. $b_j = \varepsilon a_{ji}$ ie $\min_j \frac{b_j}{a_{ji}}$

What about the objective fcn

$$c_B^T (b - \varepsilon A_i) + \varepsilon c_i$$

$$c_B^T b - \varepsilon c_B^T A_i + \varepsilon c_i$$

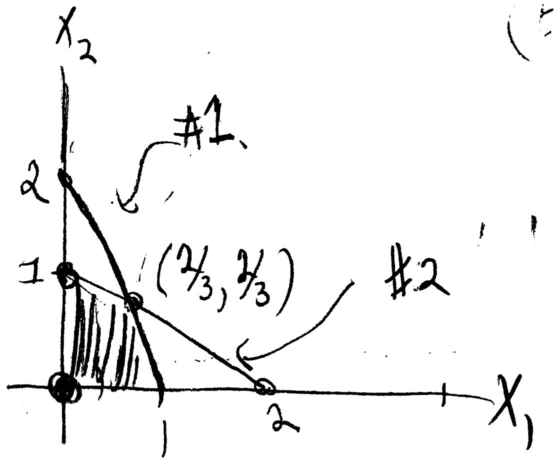
$$c_B^T b + \varepsilon (c_i - c_B^T A_i)$$

need $c_i > c_B^T A_i$ to increase $c^T x$.

$\max\ x_1 + 3x_2 \quad \begin{pmatrix}1\\3\end{pmatrix}^T \begin{pmatrix}x_1\\x_2\end{pmatrix}$

$2x_1 + x_2 \le 2$

$x_1 + 2x_2 \le 2 \qquad x_1, x_2 \ge 0$



$$\max \begin{pmatrix}1\\3\end{pmatrix}^T \begin{pmatrix}x_1\\x_2\end{pmatrix} \text{ sub } \begin{pmatrix}2 & 1\\1 & 2\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix} \le \begin{pmatrix}2\\2\end{pmatrix}$$

try BS.

$$\begin{pmatrix}1\\3\end{pmatrix}^T \begin{pmatrix}0\\1\end{pmatrix} = 3$$

$$\begin{pmatrix}1\\3\end{pmatrix}^T \begin{pmatrix}2/3\\2/3\end{pmatrix} = \frac{2}{3} + 3(2/3) = 2\tfrac{2}{3}$$

Slack form

$$\begin{pmatrix}2 & 1 & 1 & 0\\1 & 2 & 0 & 1\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix} = \begin{pmatrix}2\\2\end{pmatrix} \qquad x_1, x_2, x_3, x_4 \ge 0$$

$$B = \{1, 2\} \qquad A_B = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$$

$$\begin{array}{cccc} x_3 & x_4 & x_1 & x_2 \end{array}$$

$$\begin{pmatrix}1 & 0 & 2 & 1\\0 & 1 & 1 & 2\end{pmatrix}\begin{pmatrix}x_3\\x_4\\0\\0\end{pmatrix} = \begin{pmatrix}2\\2\end{pmatrix}$$

$$\begin{array}{cc} 0 & 0 \end{array}$$

$c^T A_1$

$$\bar{x} = \begin{pmatrix}0\\0\\2\\2\end{pmatrix} \text{ bfs}$$

$$\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}x_3\\x_4\end{pmatrix} = \begin{pmatrix}2\\2\end{pmatrix} \Rightarrow \begin{matrix}x_3 = 2\\x_4 = 2\end{matrix}$$

Goal:

Given bfs $\bar{x} = \begin{pmatrix} 0 \\ 0 \\ 2 \\ 2 \end{pmatrix}$ find new bfs with larger $c^T x$.

Consider dropping constraint $x_1 = 0$.

objective fcn $c_B = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $c_B^T A_1 = 0$

$c_1 = 1$ $\therefore$ $1 - 0 > 0$

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which j $b = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$ $A_i = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

$$x_B = \begin{pmatrix} 2 \\ 2 \end{pmatrix} - \varepsilon \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 - \varepsilon 2 \\ 2 - \varepsilon \end{pmatrix}$$

$j = 1$ constraint 1 ie $x_3 = 0$

$B = (2, 3)$

$$\begin{pmatrix} 2 & 0 & 1 & 1 \\ 1 & 1 & 2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_4 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

$x_1 = 1$ $x_4 = 1$