Resistive Model of a Graph & Random Walks

Motivation: Making a recommendation (NETFLIX)

Question: Should we recommend M to V? Score(V, M)

Idea 1

\[ \text{Score}(V, M) = \text{graph dist from } V \text{ to } M \]

\[ W_{ij} = \frac{1}{\text{rank}ij} \]

\[ \text{Score}(V, M) = \min_{VPM} \text{W}(P) \]

Idea 2

\[ \text{W}(P) = \min_{e \in P} (\text{rank}(e)) \]

\[ \text{Score}(V, M) = \max_{VPM} \text{W}(P) \]
Problem For 1) and 2) extra paths do not improve score

Idea 3 Score(\(V, M\)) \(\equiv\) Max flow from \(V\) to \(M\).

Problem Shorter paths do not improve score

Idea 4 View edges as conductors

\[\text{Score}(V, M) = \text{effective conductance}\]

Idea 5 Consider random walk from \(V\) to \(M\)

\[\text{Score}(V, M) = \text{hit}(V, M) + \text{hit}(M, V)\]

\[\text{hit}(V, M) \equiv \text{Expect length random walk from } V, M\]

We show 4) & 5) are related.
Resistance Theory

Ohm's Law:

\[ V = I \times R \]

- \( C \equiv \) conductance
- \( R \equiv \) resistance
- \( V \equiv \) voltage
- \( I \equiv \) current

\[ C = \frac{1}{R}, \quad I = C \cdot V = \frac{V}{R} \]

Facts no proof

Resistors in series:

\[ R = R_1 + \cdots + R_m \]

\[ C = \frac{1}{(\frac{1}{C_1} + \cdots + \frac{1}{C_m})} = \theta \]

i.e. \( I = \frac{V}{R} \)
Conductors in Parallel

\[ C = C_1 + \cdots + C_m \]

\[ \text{ie } i = V \cdot C \]

---

Effective Resistance/Conductance

Let \( G \) be a network of resistors

\[ V_{ab} = \text{voltage} \]

\[ i_{ab} = \text{current} \]

\[ R_{ab} = \frac{V_{ab}}{i_{ab}} \]

\[ C_{ab} = \frac{1}{R_{ab}} \]
HW) Show that $R_{ab}$ is a metric space

ie  
1) $R_{ab} \geq 0$
2) $R_{ab} = 0$ iff $a = b$
3) $R_{ab} = R_{ba}$
4) $R_{ac} \leq R_{ab} + R_{bc}$
Computing effective resistance

Use Kirchhoff's Law: \( \text{flow in} = \text{flow out} \)

An example

\[
\begin{align*}
V_1 & \quad C_1 \quad V_2 \quad V_3 \\
V & = V_0 \\
C_2 & \quad V_2 \\
C_3 & \quad V_3
\end{align*}
\]

by Ohm's Law
\[
\begin{align*}
i_1 &= C_1 (V - V_1) \\
i_2 &= C_2 (V - V_2) \\
i_3 &= C_3 (V - V_3)
\end{align*}
\]

Residual current \( i = i_1 + i_2 + i_3 \)

by Kirchhoff
\[
i_1 + i_2 + i_3 = 0
\]

\[
C_1 (V - V_1) + C_2 (V - V_2) + C_3 (V - V_3) = 0
\]

\[
(C_1 + C_2 + C_3) V = C_1 V_1 + C_2 V_2 + C_3 V_3
\]
\[ C = C_1 + C_2 + C_3 \]
\[ CV = C_1 V_1 + C_2 V_2 + C_3 V_3 \]
\[ V = \frac{C_1}{C} V_1 + \frac{C_2}{C} V_2 + \frac{C_3}{C} V_3 \]

*V is convex combination of \( V_1, V_2, V_3 \)*

**residual current** = \( CV - C_1 V_1 - C_2 V_2 - C_3 V_3 \)

**The general case**

\[ G = (V, E, C) \quad C : E \rightarrow \mathbb{R}^+ \]
\[ V = \{ V_1, \ldots, V_n \} \]

\[ d(V_i) = \sum_{(i,j) \in E} C_{ij} \]

\[ A_{ii} = \begin{cases} 
C_{ii} & \text{if } (i,i) \in E \\
0 & \text{otherwise} 
\end{cases} \]
Laplacian \( (G) = L(G) = L \)

\[ L_{ij} = \begin{cases} 
    d(v_i) & \text{if } i = j \\
    -C_{ij} & \text{if } (i,j) \in E \\
    0 & \text{otherwise}
\end{cases} \]

ie \( L = D - A \) where \( D = \begin{pmatrix} 
    d(v_1) & 0 & \cdots & 0 \\
    0 & \ddots & \ddots & \vdots \\
    \vdots & \ddots & \ddots & 0 \\
    0 & \cdots & 0 & d(v_n)
\end{pmatrix} \)

Let \( V \) be a voltage setting of nodes

Note \( (LV)_i \) = residual current at \( V_i \)

Inverse: We inject currents and get voltages.

The net injected must be zero!
Goal: \( R_{in} \)

method 1 solve \( L \left( \begin{array}{c} 0 \\ V_1 \\ \vdots \\ V_{n-1} \\ 1 \end{array} \right) = \left( \begin{array}{c} 0 \\ c \\ 0 \\ \vdots \\ -c \end{array} \right) \) \( \star \) \( c = V/R \)

\( V = V_i \)

\( R = V_i \)

\( \star \): is called a boundary valued prob.

In our case \( V_i \) \& \( V_n \) are the bdary

\( (V_1, \ldots, V_n) \) is called harmonic

because \( V_i \in \text{interior} \Rightarrow \)

\( V_i \) is convex combination of neighbors
Maximum Principle: If $f$ is harmonic, then min & max are on bdary.

If $v_i$ interior, then $\exists$ $v_i$ & $v_j$ s.t. $v_i \leq v \leq v_j$.

Uniqueness Principle: If $f$ & $g$ are harmonic with same bdary values, then $f = g$.

If $f - g$ is harmonic with zero on bdary,

$\Rightarrow f - g = 0 \Rightarrow f = g$. 

method 2 solve $kV = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}$ Does $V$ exist?

$R_{\text{in}} = V_1 - V_n$

Another way to view the laplacian

Edge-vertex Matrix

$$\Gamma = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Orient each edge

$\Gamma = \begin{pmatrix} e_1 & e_2 & e_3 & e_4 \end{pmatrix}$
Let $c_i, \ldots, c_m$ = conductance of $e_i, \ldots, e_m$

$$C = \begin{pmatrix} c_1 & 0 \\ 0 & \vdots \\ 0 & c_m \end{pmatrix}$$

Note: $\Gamma V$ = voltage drop across each edge

$C \Gamma V$ = current flow

$\Gamma^T C \Gamma^T V$ = residual current at each vertex

Thus

$L = \Gamma^T C \Gamma$
Current & Energy/Power Dissipation

\[ \frac{C}{R} \]

\[ V \]

Newton

Energy = Force \times Distance

\[ \equiv \text{Volt} \times \text{Current} \]

\[ \equiv V \times i \]

\[ \equiv CV^2 \]

\[ \equiv i^2R \]

Network

\[ E = \frac{1}{2} \sum_{x,y} i_{xy} (V_x - V_y) \]

\[ \nabla^T L \nabla = \nabla^T \Gamma^T C \Gamma \nabla = (\Gamma \nabla)^T C (\Gamma \nabla) \]

\[ = \sum_{\text{oriented}} C_{xy} (V_x - V_y)^2 = E \]

\[ (x,y) \in \mathcal{E} \]
Define \( j_x = \sum_y j_{xy} \) (residual flow at \( x \))

Let \( W \) = any voltage settings

\( j \) = any flow from \( a \) to \( b \)

**Conservation of Energy**

\[
(W_a - W_b) j_a = \frac{1}{2} \sum_{x,y} (W_x - W_y) j_{xy}
\]

\[
\text{LHS} = \sum_{x,y} (W_x - W_y) j_{xy} = \sum_x W_x \sum_y j_{xy} - \sum_y W_y \sum_x j_{xy}
\]

\[
= W_a \sum_y j_{y} + W_b \sum_y j_{y} - (W_a \sum_x j_{x} + W_b \sum_x j_{x})
\]

\[
= W_a j_a + W_b j_b - W_a (-j_a) - W_b (-j_b)
\]

\[
= W_a j_a - W_b j_a + W_a j_a - W_b j_a = 2(W_a - W_b) j_a
\]
Suppose $a, b \in V$ effective resistance $R_{ab}$

Effective energy $\frac{1}{2} R_{ab} = R_{ab}$ if $c_{ab} = 1$

Real energy using Kirchhoff's Law

Solve $HV = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ \hspace{1cm} $V_a = V_i$ \hspace{1cm} $V_b = V_n$

Energy $\equiv V^T L V = V^T \begin{pmatrix} 1 \\ 0 \end{pmatrix} = V_a - V_b$

Thm. Real Kirchhoff energy $\equiv$ effective energy

---

**Minimum Energy Flow**

**Def.** (OR type flow)

$j : E \rightarrow \mathbb{R}$ is a flow from $a \rightarrow b$ if:

1) $j_{xy} = -j_{yx}$

2) $\sum_j j_{xy} = 0$ if $x \neq a, b$

3) $j_{xy} = 0$ \hspace{1cm} $(x, y) \in E$
Thomson's Principle

\[ i \text{ is a unit Kirchhoff flow from } a \text{ to } b \]
\[ j \text{ is any unit flow from } a \text{ to } b \]

then \[ \sum i_{xy}^2 R_{xy} \leq \sum i_{xy}^2 R_{xy} \]

So let \( d = j - i \) then \( d \) is a zero flow if \( d_a = 0 \)

\[ \sum j_{xy}^2 R_{xy} = \sum (i_{xy} + d_{xy})^2 R_{xy} \]
\[ = \sum i_{xy}^2 R_{xy} + 2 \sum i_{xy} R_{xy} d_{xy} + \sum d_{xy}^2 R_{xy} \]
\[ + 2 \sum (V_x - V_y) d_{xy} \text{ (*)} \]

Set \( W = V \) & \( j = d \) then by conservation of energy

\( (*) = 4 (V_a - V_b) d_a = 0 \) thus

\[ \sum j_{xy}^2 R_{xy} = \sum i_{xy}^2 R_{xy} + \sum d_{xy}^2 R_{xy} \]
\[ \geq \sum i_{xy}^2 R_{xy} \]
Rayleigh's Monotonicity Law

If \( \forall x, y \quad R_{xy} \geq R_{xy} \) then \( \bar{E} R_{ab} \geq E R_{ab} \)

Let \( j \) = unit flow from \( a \) to \( b \) in \( \bar{R}_{x} \)
\( i = "R" \)

\[
\bar{E} R_{ab} = \int \bar{E} R_{ab} = \frac{1}{2} \Sigma j_{xy}^{2} \bar{R}_{xy} \\
\geq \frac{1}{2} \Sigma j_{xy}^{2} R_{xy} \\
\geq \frac{1}{2} \Sigma i_{xy}^{2} R_{xy} \quad (Thomson) \\
= E R_{ab}
\]