Random Walks on Graphs

Graph $G = (V, E, w)$ (possibly directed)

$w: E \rightarrow \mathbb{R}^+$

$w_i = w(v_i) = \sum_{ij \in E} w_{ij}$  
$P_{ij} = w_{ij} / w_i$

Random walk on $G$

Suppose, at a given time, we are at $v_i \in V$.

We move to $v_j$ with probability $P_{ij}$

Eg. $V$ = all permutations of a deck of cards  
$P_{ij}$ = prop of going from perm i to perm j in one shuffle.

Why do professional players play from a deck after 5 shuffles?
Important Parameters

Access time or Hitting time

\[ H_{ij} = \text{Expected time to visit } j \text{ starting at } i \]

Commute Time

\[ K(i,j) = H(i,j) + H(j,i) \]

Cover Time

Expected time to visit all nodes max over all starting nodes

Mixing Rate (not covered)
Random walks - the Symmetric Case

Do a random walk on a network of conductors!

Input: $G = (V, E, C)$, $C_{ij} = C_{ji}$, $a, b \in V$

Consider a random walk starting at $x$ and ending at $b$.

**Def:** $h_x = \text{prob we visit } a \text{ before visiting } b$. $a \neq b$ starting from $x$.

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  1 2 1 3 1 4 1 5
1 2 1 3 4 1 5
G
a  x  b
```

$h_a = 1$, $h_b = 0$

$h_a > \frac{1}{2}$? why?
\[ h_a = 1 \quad \& \quad h_b = 0 \]

Suppose \( X \neq a, b \)

Claim \( h_x = \sum_y P_{xy} h_y \)

\( P_{xy} \geq 0 \quad \sum_y P_{xy} = 1 \)

\( h_x \) is a convex combination of its neighbors!

\( h \) is harmonic with boundary \( a, b \)!

Let's solve the electrical prob

\( V_a = 1 \quad \& \quad V_b = 0 \quad \text{and} \quad X \neq a, b \) float.

\( x \neq a, b \) \( V_x = \sum_y \frac{C_{xy}}{C_x} V_y \) but \( \frac{C_{xy}}{C_x} = P_{xy} \)

\( \Rightarrow h = V \)

Thm \( V_a = 1 \quad \& \quad V_b = 0 \) then \( V_x = \) prob visit \( a \) before \( b \).
\[ h_c = \frac{3}{4} \]

What does it mean (in random walks) if we set \( V_{a_1} = V_{a_2} = 1 \) & \( V_{b_1} = V_{b_2} = 0 \)?

\( X \neq a_1, a_2, b_1, b_2 \) float?
Interpretation of Current

Assume \( G = (V, E, C) \), \( a, b \in V \)

Consider 1 unit of current flow from \( a \) to \( b \),

Say \( i \)

What does \( i_{xy} \) correspond to in random walks?

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\[ i_{xy} = \text{Expected net # of traversals of } E_{xy} \text{ in random walk from } a \text{ to } b. \]

pf Slides 7, 8, 8A
Let's start with:

\[ U_x = \text{Expected number of visits to } X \text{ before reaching } b \text{ starting at } a. \]

\[ U_b = 0, \quad x \neq b \]

\[ U_x = \sum_y U_y P_{yx} \quad \text{note } \sum_y P_{yx} \neq 1 \]

Recall \( C_x = \sum_y C_{xy} \)

\[ \text{note } \quad C_x P_{xy} = C_x \left( \frac{C_{xy}}{C_x} \right) = C_{xy} = C_{yx} = C_y \left( \frac{C_{yx}}{C_y} \right) = C_{xy} P_{yx} \]

\[ U_x = \sum_y U_y \frac{C_y P_{yx}}{C_y} = \sum_y U_y \left( \frac{P_{xy} C_x}{C_y} \right) \]

\[ \frac{U_x}{C_x} = \sum_y P_{xy} \left( \frac{U_y}{C_y} \right) \]
Let \( V_x = \left( \frac{U_x}{C_x} \right) \) then

\[ V_x = \sum_y P_{xy} V_y \quad V_x \text{ is harmonic!} \]

What is the boundary? \( V_b = 0 \)

Suppose we know \( U_a \) \( V_a = U_a / C_a \)

\( V \) is a voltage where \( V_b = 0 \) & \( V_a = U_a / C_a \)

Let \( J_{xy} \) be its current

\[ J_{xy} = (V_x - V_y) C_{xy} = \left( \frac{U_x}{C_x} - \frac{U_y}{C_y} \right) C_{xy} \]

\[ = U_x \left( \frac{C_{xy}}{C_x} \right) - U_y \left( \frac{C_{yx}}{C_y} \right) = U_x P_{xy} - U_y P_{yx} \]
\[ U_x P_{xy} = \text{expected \# of traversals from } X \text{ to } Y \]
\[ U_y P_{yx} = \text{"y to x"} \]
\[ I_{xy} = \text{expected \# net } xy \text{ traversals} \]

What is net current flow from a to b?

\[ \text{ie } \sum_{y} I_{xy} \]

This must be 1

This proves Thm
How to compute hitting time

\[ h(x;b) = \text{expected time to reach } b \text{ from } x \]

\[ h_x = h(x;b) \quad b \text{ fixed} \]

Let's write a recurrence:

\[ h_b = 0 \quad x \neq b \]

\[ h_x = 1 + \sum_y h_y P_{xy} \quad (\star) \]

How do we solve (\star)?

Let's think of \( h_x \) as a voltage \( V_x \)

\[ V_b = 0 \quad V_x = 1 + \sum_y \frac{C_{xy}}{C_x} V_y \]
\[ C_x V_x = C_x + \sum_Y C_{xy} V_y \]

\[
\frac{C_x V_x - \sum_Y C_{xy} V_y}{Y} = C_x
\]

- Graph Laplacian
- Residual current

\[
L \mathbf{V} = \begin{pmatrix}
C_x \\
\vdots \\
C_{n-1} \\
\delta
\end{pmatrix} \quad C = \sum C_i
\]

\[
V_n = 0 \quad b = V_n
\]

by conservation of flow

\[
\delta = C_n - C
\]

Alg for hitting time

Solve \( L \mathbf{V} = \begin{pmatrix}
C_x \\
\vdots \\
C_{n-1} \\
C_{n-C}
\end{pmatrix} \)

return \( V_x \)
What about commute time?

\[ a = V_i \quad \& \quad b = V_n \]

**Solution 1**

Solve

\[ LV^b = \begin{pmatrix} \mathbf{C}_n \\ \mathbf{1} \\ \mathbf{C}_n - \mathbf{C} \end{pmatrix} \quad LV^a = \begin{pmatrix} \mathbf{C}_1 - \mathbf{C} \\ \mathbf{1} \\ \mathbf{C}_n \end{pmatrix} \]

\[ h(1,n) = V^b_i - V^b_n \]
\[ h(n,1) = V^a_n - V^a_i \]
\[ V = V^b_i - V^a_i \]

\[ C(1,n) = (V^b_i - V^a_i)_1 - (V^b_n - V^a_n)_n \]

**Solution 2**

\[ L(V^b - V^a) = LV^b - LV^a = \begin{pmatrix} \mathbf{C}_1 \\ \mathbf{1} \\ \mathbf{C}_n - \mathbf{C} \end{pmatrix} - \begin{pmatrix} \mathbf{C}_1 - \mathbf{C} \\ \mathbf{1} \\ \mathbf{C}_n \end{pmatrix} = \begin{pmatrix} \mathbf{C}_1 \\ \mathbf{1} \\ \mathbf{0} \end{pmatrix} = \mathbf{C} \begin{pmatrix} \mathbf{1} \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix} \]
solve $LV = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

return $C(V_1 - V_n)$ but $(V_1 - V_n) = R_{in}$

Thm $C(a, b) = BR_{ab} \cdot C$