The Preflow-Push Max-Flow Alg

Goal: $O(n^3)$ Alg

This Alg + Better Data Structures: $O(n \cdot m \log n)$

Known: $O(m\sqrt{n} \log^k n)$ (some k)

Idea Relax the flow-in = flow-out condition.

Def f is an s-t preflow (preflow) if

1) $f: E \to R^+$
2) $0 \le f(e) \le C_e$
3) $e_f(v) = \sum_u f(u,v) - \sum_u f(v,u) \ge 0$

As before we have a residual graph.

G_f

We use a vertex label to guide "pushing" the flow.

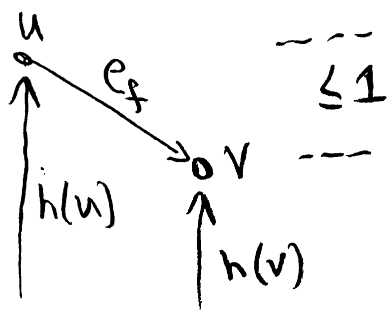
Let $h: V \to \mathbb{Z}^+ = \{0, 1, 2, \cdots\}$ (label or height)

<u>Def</u> f & h are <u>compatible</u> if

1) $h(t) = 0$ & $h(s) = n$ (sink & source condition)

2) $\forall (u,v) \in E_f \quad h(u) \le h(v) + 1$

eg



<u>Lemma</u> f (st preflow) & h (compatible labeling) then no augmenting path in G_f.

pf By contradiction! Let P be such a path

We may assume P is simple. # edges of $P < n$

$\therefore h(s) - h(t) < n$ contra!

<u>Cor</u> f (s-t flow) & h (compatible labeling)

then f is a Max-Flow.

Init(G) $f(s,v) = C(s,v) \quad \forall v$

$f(e) = 0$ o.w.

$h(s) = n \quad h(v) = 0$ o.w.

<u>Note</u> f & h are compatible

<u>pf</u> Only v st $h(v) > 0$ is s.

But we saturated edges out of s.

Push & Relabel

Push (f, h, v, w)

if $e_f(v) > 0$ & $(v,w) \in E_f$ & $h(v) > h(w)$

then "push" $\min\{ e_f(v) ; c_f \}$

eg

$e_f(v) = 1/2$

$h(v) = 10$ & $h(w) = 9$

Relabel (f, h, v)

If $e_f(v) > 0$ & $\forall (v,w) \in E_f$ $h(v) \leq h(w)$

then $h(v) \leftarrow h(v) + 1$

Preflow-Push

1) Init
2) While $\exists v \neq t$ st $e_f(v) > 0$ do
   If $\exists w$ st $(v,w) \in E_f$ & $h(v) > h(w)$
   then Push(f,h,v,w)
   Else relabel(f,h,v)
3) Return f

Correctness

Claim1 f & h are always compatible.

Claim2 If Preflow-Push returns f then f is a max-flow

pf of claim 1

Each push preserves compatiblity.
" " relabel " ".

Timing & Termination

To show excess can be "pushed" back to s.

Lemma f preflow & $e_f(v) > 0$ then $\exists$ path from v to s. (s not t)

pf Let $A = \{ w \mid \exists \text{path from } w \text{ to } s \}$

$B = V - A$

Note Flow can only go from B to A.

Claim $\sum_{v \in B} e_f(v) = 0$

$$\sum_{v \in B} e_f(v) = \sum_{v \in B} f^{in}(v) - f^{out}(v)$$

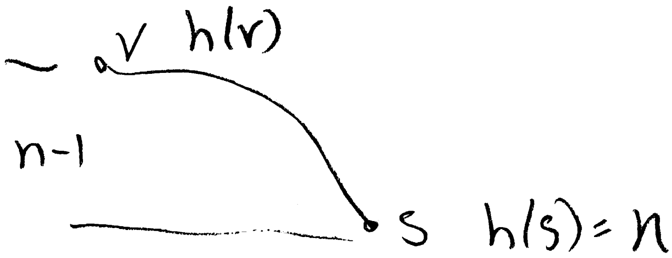
three types of edges $(u,v) \in$ Sum

1) $u, v \in B$ (cancel)
2) $u \in A$ & $v \in B$ (zero)
3) $v \in B$ $u \in A$ (negative in sum)

i.e. $0 \le \sum_{v \in B} e_f(v) = -f^{out}(B) \le 0$ QED

Claim $h(v) \le 2n-1$

pf



$\Rightarrow h(v) \ge 2n-1$

Cor # of relabels $< 2n^2$

Count # of pushes

2 types

saturating push $\equiv c_f$

nonsaturating otherwise

Claim # saturating pushes $\le 2nm$

to show # per edges $\le n$

Before push $\quad v \xrightarrow{e_f} w \quad h(v) = h(w)+1$

After push $\quad v \xleftarrow{e_f} w \quad$ need 2 relabels.

Lemma # nonsaturating pushes $\le 4n^2 m$.

pf Consider the following potential

$$\Phi(f,h) = \sum_{e_f(v)>0} h(v)$$

ops	#	AC	Total	unit-cost
relabel	$2n^2$	≤ 2	$4n^2$	1
saturating	$2nm$	$2n-1$	$2nm(2n-1)$	1
non-sat	?	0	0	1

$\Rightarrow ? \le 4n^2 m$

We have shown

<u>Thm</u> Preflow-Push is $O(n^2 m)$ alg.

<u>New alg</u> Preflow-Push-Max-Height

Let $H = \max\limits_{e_f(v)>0} h(v)$

Alg Push(v) st $e_f(v) > 0$ & $h(v) = H$

<u>Thm</u> Preflow-Push-Max-Height is $O(n^3)$

$\Phi(f,h) = H$

Note: Only relabel increases Φ.

$2n^2$ increases and $4n^2$ total changes.

<u>Goal:</u> # of non-sat pushes per change.

at most one per node or n.

# of non-sat pushes is $\leq 4n^3$.