Parallel Algorithms

15-750
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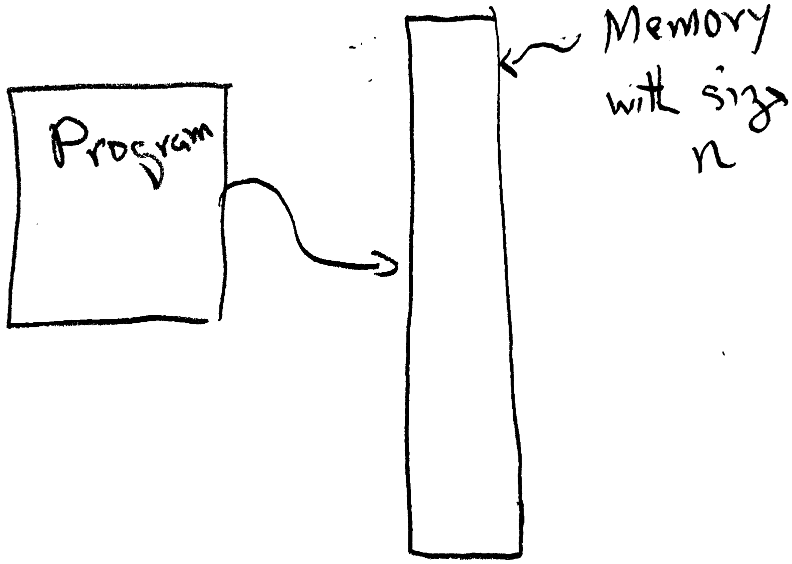
So far we have assumed the following

model: RAM model



unit time ops

(+, ×, ÷ (log n bits))

(read/write into memory)

A central model to describe Graph Algorithms.

Other models:

1) Agents / ants 2) pointer machines

RAM is unrealistic as n goes to infinity.

1) speed of light (large size machines)
2) quantum effects (small size machines)

Bottom Line: RAM

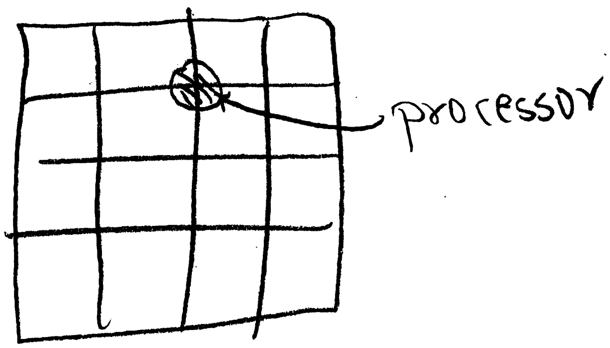
1) Many important algorithms were found using this model.
2) Most algorithms are coded in a RAM like language. eg C

Parallel Models

Fixed connection machines

machines ≡ 1) finite state machine
2) RAM

A) Cellular Arrays 1D, 2D, 3D



2D

1940's von Neumann

60's, 70's algorithms for CA.

Alvy Ray Smith 1974

Highly connected models

1) Hypercube $\equiv (V, E)$ 1980's

$$V \equiv \{(a_1, \dots, a_m) \mid a_i \in \{0,1\}\} \quad m = \log n$$

$$((a_1 \cdots a_m), (a_1 \cdots \bar{a}_i \cdots a_m)) \in E$$

2) Shuffle-exchange graph 1980's

$$V = \{(a_1 \cdots a_m) \mid a_i \in \{0,1\}\}$$

$$((a_1 \cdots a_m), (\bar{a}_1 a_2 \cdots a_m)) \in E$$

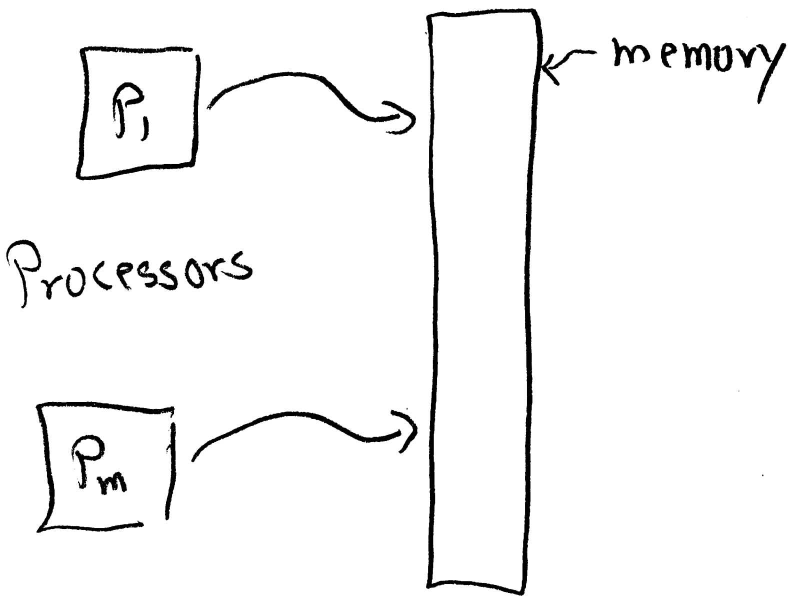
$$((a_1 \cdots a_m), (a_m a_1 \cdots a_{m-1})) \in E$$

3) Randomly connected graphs

possible models of the brain!

Shared memory models

1) PRAM (Parallel Random Access Machine)



	Exclusive	Concurrent
Read		
Write		

Unit time ops

$+, \times, \div$

read / write

ER EW

CR CW

PRAM issues:

Penalty for CR on an ER machine?

1) Machine crashes!

2) garbage read!

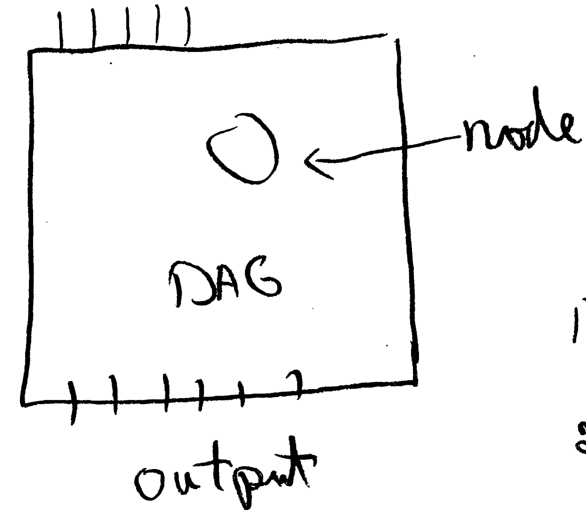
How is synchronization handled?

1) After each unit of time!

2) none!

Circuit Model

inputs in either bits or words



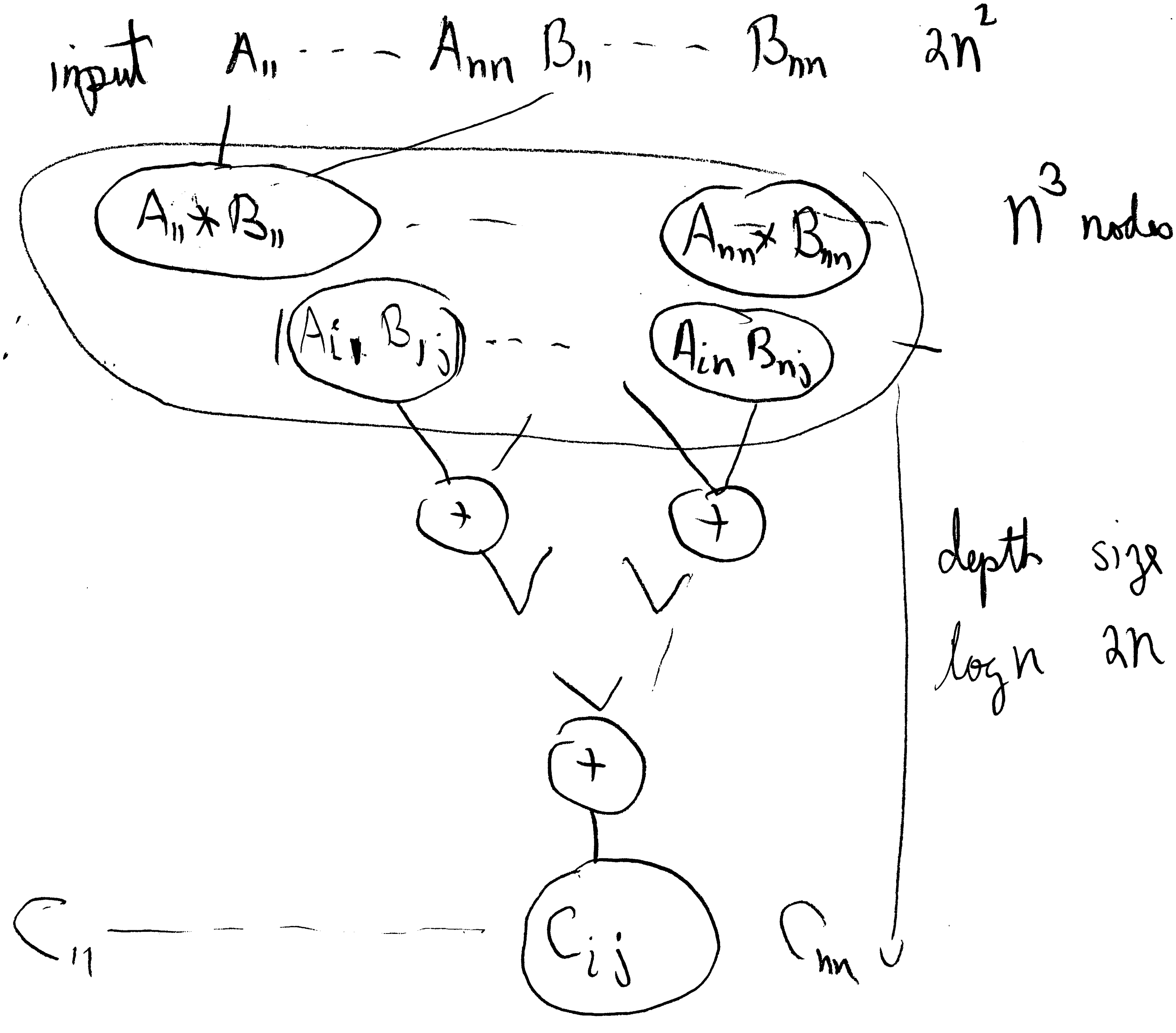
nodes 1) $\wedge, \vee, \neg$ gates
2) Arithmetic ops

1) Constant fan in.
2) Arbitray fan out.

Work $\equiv$ # nodes
Time $\equiv$ Longest path from input to output

Naive Matrix Multiply in the Circuit Model

$$C_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj} \qquad C = A \cdot B$$



Totals: Work: $O(n^3)$

Time: $O(\log n)$

Naive MM on PRAM

$P \equiv$ # processor $T \equiv$ Parallel time

1) CRCW $P \equiv O(n^3)$ one processor / node

$T \equiv O(\log n)$

Note: fan out $\equiv$ reads

fan in $\equiv$ writes

2) CREW $P \equiv O(n^3/\log n)$

$T \equiv O(\log n)$

each process does the work of $\log n$ virtual process.

3) EREW $P = n^3/\log n$ $T \equiv O(\log n)$

Def Work $\equiv P \cdot T$
One must pay for each process for life of the run.

Naive MM: $O(n^3)$ work $O(\log n)$ time

Claim: If we have $P < n^3/\log n$ processors then the Time $= W/P$.

Strassen's Alg

Recall: recurrence

$$MM(n) = 7\, MM(n/2) + cn^2$$

7 recursive calls

matrix additions

$A+B$

<u>note</u> Matrix addition

$O(n^2)$ work

$O(1)$ time

Time: $T(n) = T(n/2) + O(1)$

parallel calls

$O(\log n)$

Work: $w(n) = 7w(n/2) + cn^2$

we must pay for each call!

$O(n^{2.81\cdots})$

All-Prefix-Sums / Prescan

Let $\oplus$ be an associative binary op

eg $+$, $\circ$

Def: All-Prefix-Sums

input: $[a_0, \dots, a_{n-1}]$

output: $[a_0, a_0 \oplus a_1, a_0 \oplus a_1 \oplus a_3, \dots, a_0 \oplus \dots \oplus a_{n-1}]$

Prescan

output $[I, a_0, a_0 \oplus a_1, \dots, a_0 \oplus \dots \oplus a_{n-2}]$

Application

Packing Memory

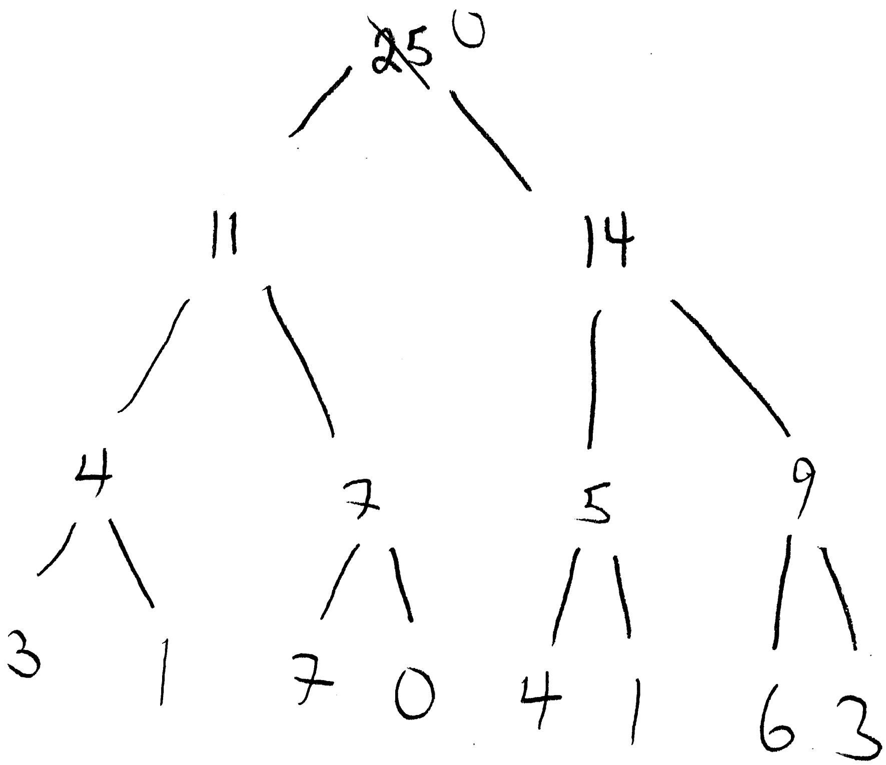
location	0	1	2	3	4	5	6	7
contents	∅	*	∅	∅	*	*	∅	*

0	1	2	3				
*	*	*	*	∅	∅		...

Prescan

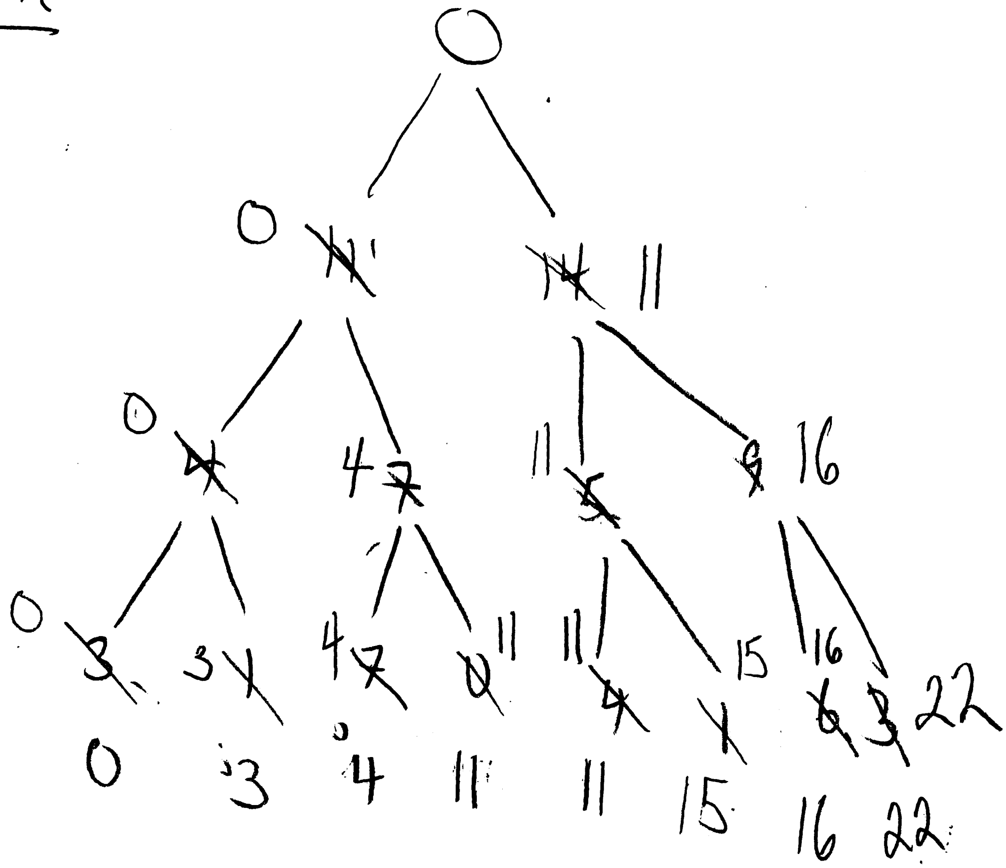
input: (3, 1, 7, 0, 4, 1, 6, 3) $\oplus$ addition

Alg 1) Compute tree of partial sums
2) set root to zero
3) DOWN!



3) Down!
a) Right-Child $\Leftarrow$ Parent $\oplus$ Left-Child
b) Left-Child $\Leftarrow$ Parent

Down



$T(n) = O(\lg n)$

$W(n) = O(n)$?

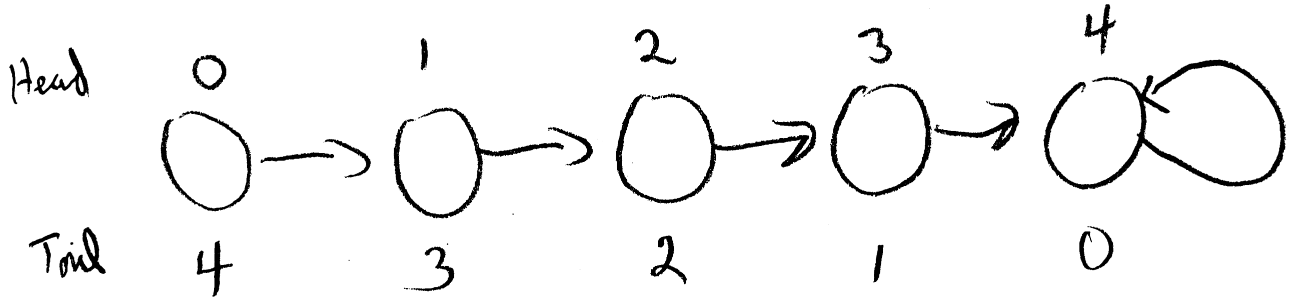
List Ranking

input: linked-list

output: a mark on each node s.t.

mark = distance from head or

= distance to tail



Assume!

1) pointers are in consecutive memory
2) We know location of head & tail
3) pointers in arbitrary order.

Wyllie's Alg

In|| $rank(!) := 1$; $rank(tail) = 0$

In|| **while** $succ(head) \neq nil$ **do**

  if $succ(!) \neq nil$ **do**

    $rank(!) := rank(!) \oplus rank(succ(!))$

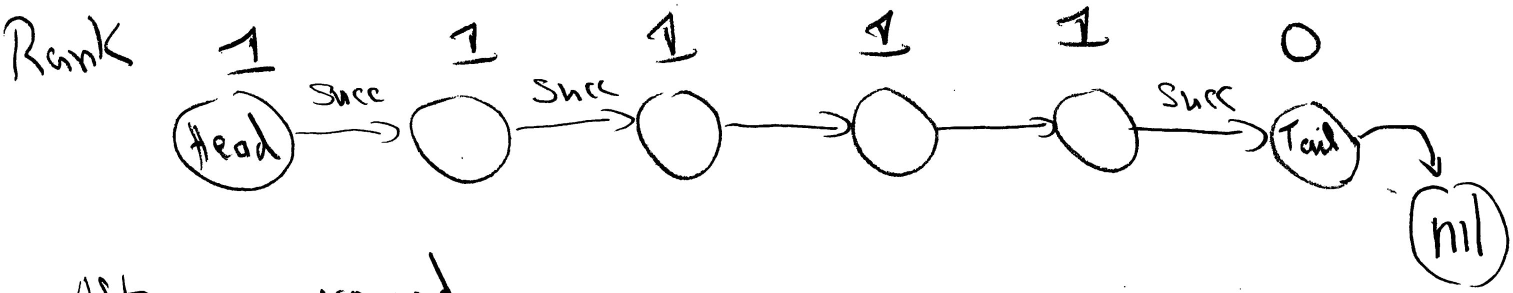
    $succ(!) := succ(succ(!))$

  fi

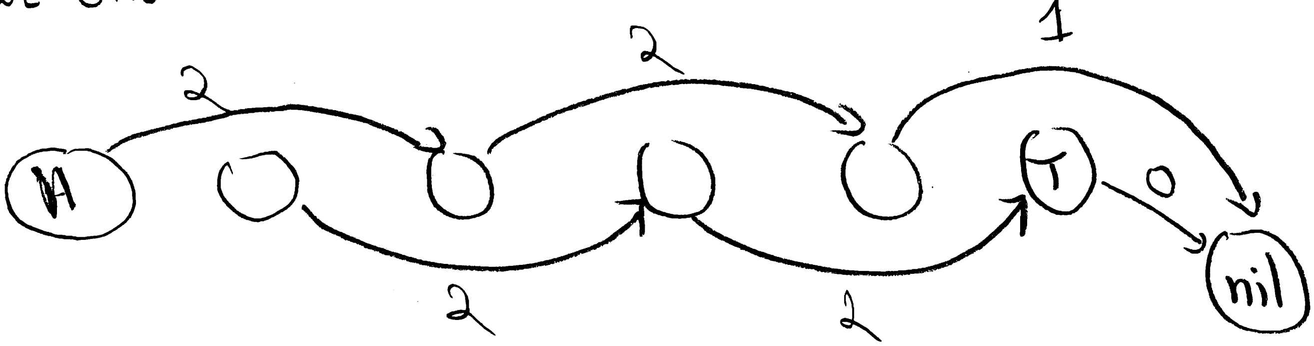
# processors = n	Goal
Time = $O(\log n)$	$O(\log n)$
Work = $O(n \log n)$	$O(n)$
CREW model	

Wyllie's Alg

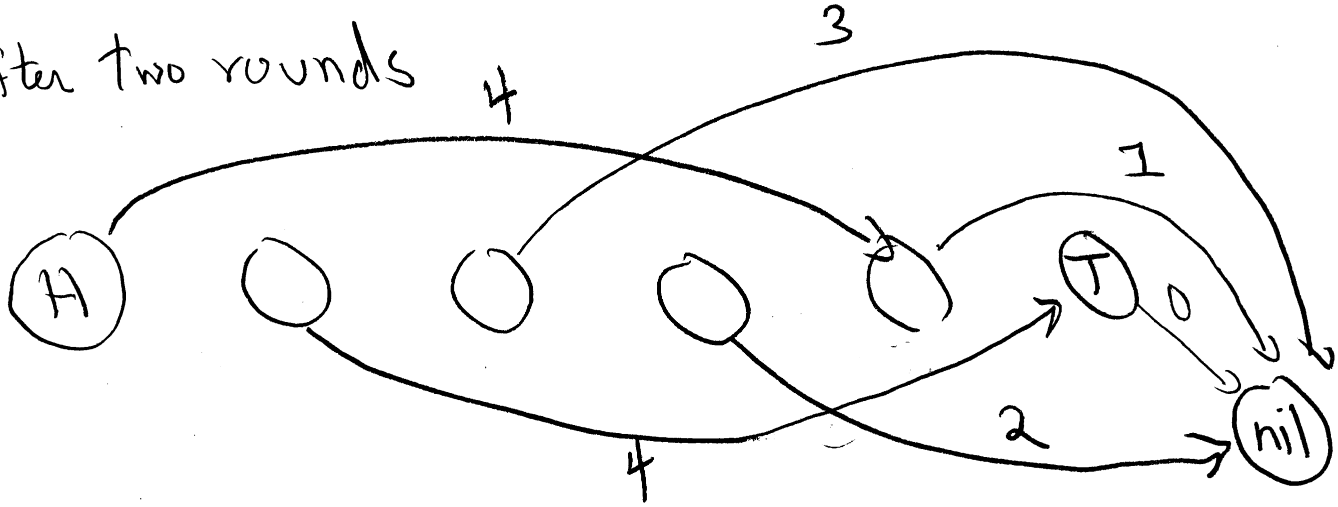
Initial



After one round



After two rounds



Random - Mate

1) Contraction Phase

1) Each live node randomly picks a sex

2) If $(F) \xrightarrow{a} (M) \xrightarrow{b} (X)$ then

$$\begin{tikzcd} (F) \arrow[rr, bend left, "a+b"] & (M) & (X) \end{tikzcd}$$

dies

3) Stop when head points to nil.
(only head is live)

<u>Thm</u> The contraction phase stops in $c \log n$ rounds with high prob.

<u>pf</u> Let $P_i \equiv$ Event that node i is still live after one round

<u>Note</u> node_i not head then $\text{Prob}[P_i] = 3/4$

Let $P_i^k \equiv$ Event that node i still live after k rounds.

Note: $\text{Prob}[P_i^k] = (3/4)^k$ i not head.

Set $k = \lfloor c \log_{4/3} n \rfloor$

$$\text{Prob}[P_i^k] = 1/(4/3)^k \leq \frac{1}{(4/3)^{c \log_{4/3} n}} = \frac{1}{n^c}$$

Let $P^k \equiv$ Event that some non-head node is still live.

assume that node_0 is the head.

$$P^k = P_1^k \cup P_2^k \cup \cdots \cup P_n^k$$

$$\text{Prob}[P^k] = \text{Prob}[P_1^k \cup \cdots \cup P_n^k]$$

$$\leq \text{Prob}[P_1^k] + \cdots + \text{Prob}[P_n^k]$$

$$\leq n \cdot \frac{1}{n^c} = \frac{1}{n^{c-1}}$$

If we set $c=2$ then the contraction phase stops with prob $\leq 1/n$.

In the expansion phase we run contraction phase "backwards".

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