Packing out memory

Input: Array of used & unused memory

Output: Array of used followed by unused memory

Input: \[ A \phi A B D C \phi \phi D \phi A B \]

Output: \[ A A B D C D A B \phi \phi \phi \phi \phi \]

Solution: Use PreScan on array

\[ M_i = \begin{cases} 1 & \text{if } i \text{th memory used} \\ 0 & \text{otherwise} \end{cases} \]
**Euler Tours**

Problem: Preorder number a tree

Input: A tree stored using pointers

Output: Preorder numbering

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**Weighted List Ranking**

Input: 

Output: 

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Diagram: A weighted tree with nodes labeled 0, 1, 2, 3, 4, 5, and edges labeled with weights.
$I$ = weight of down edge
$O$ = weight of up edge

Inorder

Postorder?
Parallel Expression Evaluation

Example Input:

```
+   
/  \ 
X   7
+   X
/ \ / \\
1 4 5 3
```

Output: Value, all subvalues

Goal: Parallel Alg

Simple Alg

1) Assign a processor to each node.
While tree non empty do
2) if leaf "send" value to parent
   delete node [RAKE]
3) if node has 2 values then evaluate.
Worst Case for simple Alg

Recall: Horner's Rule

Input: polynomial \( a_0 + a_1X + a_2X^2 + \ldots + a_nX^n \)

Alg: \( a_0 + X(a_1 + X(\ldots + X(a_{n-1} + X(a_n)))) \)

As a tree

Simple Alg

\( O(n) \) Time

\( O(n^2) \) Work
Keeping nodes with only one value busy!

Here we view the tree edges as transformers.

Init: The edge are the identity.

\[ a \circ y = ay + ab \]
The general case.

\[ f(y) = ay + b \quad g(y) = cy + d \]

\[ f(g(y)) = a( cy + d ) + b = acy + (ad + b) \]

Note: forms \( ay + b \) are closed under compositions.

We can also remove an independent set of 1-child nodes (degree 2 nodes).

Very similar to pivoting in Gaussian Elim.
Def \( V_0 \cdots V_K \) is a chain if:

1) \( V_{i+1} \) is only child of \( V_i \), \( 0 \leq i < K \).
2) \( V_K \) has only one child & it is not a leaf.

The Independent Set

1) All leaves
2) Max independent set from each maximal chain
Parallel Tree Contraction

RAKE = remove all leaves

COMPRESS = replace each maximal chain of length $K$ with one of length $\frac{K}{2}$.

\[ \text{CONTRACT} = \{ \text{RAKE}, \text{COMPRESS} \} \]

**Thm** \[ |\text{CONTRACT}(T)| \leq \frac{2}{3} |T| \]

**pf**

\[ \text{Def } V_0 = \text{leaves of } T \]

\[ V_1 \subseteq V \text{ with 1 child} \]

\[ V_2 \subseteq V \text{ with 2} \leq \# \text{ children} \]

\[ C \subseteq V_1 \text{ with child in } V_0 \]
Claim \[ |V_0| > |V_a| \]

Pf induct on size of \( T \).

Claim: \[ |V_0| > |C| \]

Def \[ R_a = V_0 \cup V_a \cup C \]

\[ C_{om} = V_i - R_a \]

\[ \text{RAKE}(R_a) \subseteq V_a \cup C \Rightarrow |\text{RAKE}(R_a)| \leq \frac{2}{3} |R_a| \]

Note \( C_{om} \equiv \text{union of maximal chains} \)

\[ |\text{COMPRESS}(C_{om})| \leq \frac{1}{a} |C_{om}| \]

Cor After \( \log_{\frac{a}{2}} n \) CONTRACTS \( T \in \) empty
A Simple Randomized List-Ranking

Assume linked-list is doubly linked.

Alg Splicing-Out
1) Make \( \frac{\log n}{\text{queue/proc}} \) queues of size \( \log n \) (\( \text{queue/proc} \))
2) Set sex of all nodes to M
3) Reset sex of each queue-top to random sex.
4) If top is F and points to M then "splice-out" top.
5) Repeat while some queue not empty (3); 4)

Thm After \( O(\log n) \) rounds all queues are empty with high probability.
Chernoff Bounds

Let $X_1, \ldots, X_t$ be independent $0/1$ random variables.

Assume $\text{Prob}(X_i = 1) = p$.

The binomial random variable is

$$S^p_n = X_1 + \ldots + X_t$$

$$\text{Expect}(S^p_t) = \sum \text{E}(X_i) = p \cdot t$$
\[ \text{Thm} \quad \Pr(\mathcal{S}_t^p < (1-\beta)p^t) < e^{-\beta^2 pt/2} \quad \forall 0 \leq \beta < 1 \]

\[ \text{Thm} \quad \Pr(\mathcal{S}_t^p > (1+\beta)p^t) < e^{-\beta^2 pt/3} \quad \forall 0 \leq \beta \leq 1 \]
Let's fix one of the queues, say $Q$.

At a given round the prob Top is spliced-out is $\geq \frac{1}{4}$

View prob as:

We have a coin $\text{Prob(Head)} = \frac{1}{4}$ $\text{Prob(Tail)} = \frac{3}{4}$

Question: After $t$ flips what is $\text{Prob}[\#\text{heads} < \log n]$?

Suppose we pick $t$ s.t. $\text{Expect \#heads} = 4 \log n$

i.e $t = 16 \log n$
We apply Chernoff with $p = \frac{1}{4}, t = 16 \log n, \beta = \frac{3}{4}$

\[
\text{Prob}\left(S_t^p < (1-\beta)pt\right) < e^{-\beta^2 pt/2}
\]

\[
\text{Prob}\left(S_t^p < \log n\right) < e^{-(\frac{3}{4})^2(\frac{1}{4})16(\log n)^{-1/2}}
\]

\[
= e^{-9/8 \log n} = n^{-9/8}
\]

Thus, Prob that some queue is not empty after $t = 16 \log n$ rounds is

\[
< (\frac{1}{4})^{1/2} n^{-9/8} < n^{-1/8}
\]