More NP-Completeness

Graph Coloring

Throughout $G = (V, E)$ undirected

Def A $k$-coloring is a func $C : V \rightarrow \{0, 1, \ldots, k-1\}$

s.t. the end-points of each edge have distinct colors.

Def 3-Color $= \{ G \mid G$ is 3-colorable $\}$

Thm 3-Color is NP-Complete

the color gadget:

$C = 3$-coloring
Colors $= \{0, 1, 2\}$
**Critical Facts**

1) \( \forall C \; \text{if} \; C(a) = C(b) \; \text{then} \; C(a) = C(c) \)

\[ \text{pf} \; \text{WLOG} \; C(a) = C(b) = 0 \]

Thus

2) \( \forall x \neq y, \; z \in \{0, 1, 2\} \; \exists C \; \text{st} \)

\[ C(a) = x \land C(b) = y \land C(c) = z \]

WLOG \( x = 0 \land y = 1 \)

WLOG \( z \in \{0, 2\} \)
Claim: \( 3 \text{CNF} \leq_p 3\text{-Color} \)

The construction of \( f(q) \)

\( q \) has clauses \( C_1, \ldots, C_k \)

Variables \( x_1, \ldots, x_n \)

A color wheel for each variable

Colors = \( \{O, T, F\} \)

Our color wheels

WLOG \( C(a) = O \) & \( C(b) = F \)
Attaching 3-Clansos to color wheels

\[ C_j = (x_i \lor x_i' \lor x_n) \]

Add one for each clause
Claim $\Phi$ is satisfiable if $G = f(\Phi)$ is 3-colorable

($\Rightarrow$) i) Use truth assignment to color color-wheels with colors $\{0, T, F\}$

ii) Color gadgets

$\Leftarrow$ View $C(a) = 0$ & $C(b) = F$ & remaining color as $T$.

Use colors of $x_1, \ldots, x_n$ as truth assignment.

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**Def** Planar-$k$-color = \{ $G$ planar | $G$ $k$-colorable \}

**Thm** Planar-$4$-color $\in \mathbf{P}$

**Thm** Planar-$3$-color is $\mathbf{NP}$-Complete
The gadget of all gadgets!

the Crossover gadget

1) If \( C \) is a 3-coloring then
\[ C(a) = C(b) \land C(c) = C(d) \]

2) Given \( x, y \in \{0, 1, 2\} \) \( \exists C \) s.t.
\[ x = C(a) = C(b) \land y = C(c) = C(d) \]
Given $G=(V,E)$ Goal construct $f(G)$

1) "Draw" $G$ in plane with crossings
2) Insert a crossover gadget for each crossing

$G = \quad f(G) = \quad $
Exact Cover

Input: finite set \( X \), subsets \( \{S_1, \ldots, S_k\} = \mathcal{S} \)

Question: \( \exists S' \subseteq \mathcal{S} \) s.t. \( S' \) is an exact cover in \( X \)\( \setminus \bigcup \mathcal{S} \)

Thm: Exact Cover is \( \text{NP} \)-Complete

Claim: \( 3\text{-Color} \leq_p \text{Exact Cover} \)

\( G = (V, E) \) is 3-color problem & \( C = \{ \text{red}, \text{blue}, \text{green} \} \)

Def: \( N(u) = \{ v \in V \mid (u, v) \in E \} \) for \( u \in V \)
The Finite set $X$

For each $u \in V$ add four points
$u$, $P_u$, $P_u\red$, $P_u\green$

For each $(u,v) \in E$ add six points
$g\red_u\red_v$, $g\red_u\green_v$, $g\green_u\red_v$, $g\green_u\green_v$

The subsets $S$

For each $u \in V$, subsets $S_u\red$, $S_u\blue$, $S_u\green$

$S_u\red = \{ u, P_u\red \}$
$N(u) = \{ v_1, \ldots, v_d \}$

Include
$\{ u, P_u\red \}, \{ u, P_u\blue \}, \{ u, P_u\green \}$
$\{ \{ g_u, g_v \} \mid (u,v) \in E \land C \neq C' \}$
Claim: \( G \) is 3-colorable iff \( f(G) \) has an exact cover.

\((\Rightarrow)\) Suppose \( C: V \to \{\text{red, blue, green}\} \)

Proof by example! Suppose \( (u, v) \in E \) \( C(u) = \text{red} \) & \( C(v) = \text{blue} \) \( d(u) = 5 \) & \( d(v) = 3 \)

Include in cover
\[ \{u, p_u^{\text{red}}\} \{v, p_v^{\text{blue}}\} \]

\( S_u^{c'} \) \( c' \neq \text{red} \)

\( S_v^{c'} \) \( c' \neq \text{blue} \)

\[ \{g_{uv}^{\text{red}}, g_{vu}^{\text{blue}}\} \]

\((\Leftarrow)\) The subsets \( \{u, p_u^{c'}\} \) give a coloring.
Knap Sack

Input:
Finite set $S$, cost fn $W: S \rightarrow \mathbb{N}$
benefit fn $b: S \rightarrow \mathbb{N}$
int $W, B$

Question:
$\exists S' \subseteq S$ st
$\sum_{a \in S'} w(a) \leq W \land \sum_{a \in S'} b(a) \geq B$

Subset Sum

Input:
Finite set $S$, $W: S \rightarrow \mathbb{N}$, int $B$

Question:
$\exists S' \subseteq S$ st
$\sum_{a \in S'} w(a) = B$
Partition Prob:

Input: Finite set \( S \), \( W: S \rightarrow \mathbb{N} \)

Question: \( \exists S' \subseteq S \) s.t.

\[
\sum_{a \in S'} W(a) = \sum_{a \notin S'} W(a)
\]

\[\text{Partition} \leq_p \text{ Subset Sum}\]

\[
\text{Set } B = \frac{1}{2} \sum_{a \in S} W(a)
\]

\[\text{Partition} \leq_p \text{ Knapsack}\]

\[
\text{Set } b = W \land W = B = \frac{1}{2} \sum_{a \in S} W(a)
\]
Subset \text{Sum} \leq_p \text{Partition}

Set \{ \emptyset \} \cup S \rightarrow N \cup \emptyset

Let \( W = \sum_{a \in S} W(a) \land N \geq W \)

New elements added to \( S \) a \& b giving \( \overline{S} \) \( W(a) = N-B \) \& \( W(b) = N-W+B \)

\underline{Claim} \quad S \text{ has subset sum} \iff \overline{S} \text{ has partition}.

(\Rightarrow) \quad \text{Suppose } S' \text{ is a subset sum}

\quad then \( S' \cup \{a, b\} \) is a partition.

(\Leftarrow) \quad \text{Suppose } \overline{S}' \text{ is a partition}

\quad if \( a \in \overline{S}' \) then \( b \notin \overline{S}' \)

\quad then \( \overline{S}' \setminus \{a, b\} \) is a subset sum.
Exact Cover $\leq_P$ Subset Sum

**Proof:** Let $(X, \mathcal{S})$ be an exact-cover problem. We construct a subset sum problem $(\overline{X}, w)$.

Set $\overline{X} = \emptyset$ (Every subset is an element)

WLOG $X = \{0, 1, \cdots, m-1\}$

for $x \in X$ \#$_x = |\{A \in \mathcal{S} \mid x \in A\}|$

pick prime $p > \#_x \ \forall x \in X$

Define $w(A) = \sum_{x \in A} p^x$, $B = \sum_{x=0}^{m-1} p^x = \frac{p^{m-1}}{p-1}$

**Claim:** Exact Cover iff Subset Sum
By example

Suppose $A_1 = \{1, 3\}$, $A_2 = \{0, 3\}$

In $p$-ary notation

$\ p^0 \ p^1 \ p^2 \ p^3 \ p^4 \ \ldots \ p^{m-1}$

$w(A_1) = (0, 1, 0, 1, 0, \ldots, 0)$

$w(A_2) = (1, 0, 0, 1, 0, \ldots, 0)$

$w(A_1) + w(A_2) = (1, 1, 0, 2, 0, \ldots, 0)$

Since $#3 < p$ the 2 can not generate a carry.