NP-Completeness (cont)

15-750
4/21/10

3CNF = { $\varphi \in$ CNF | at most 3 literals per clause }
exactly " "

Thm 3CNF is NP-Complete

3CNF $\in$ NP

to show CNF $\leq_p$ 3CNF

Idea: Clause $(a \vee b \vee c \vee d)$ and new variable x.

Claim An assignment T making $(a \vee b \vee c \vee d)$ true

iff T plus assignment of x making $(a \vee b \vee x)(\bar{x} \vee c \vee d)$ True

$T : \{a, b, c, d\} \rightarrow \{T, F\}$

$T \subseteq T' : \{a, b, c, d, x\} \rightarrow \{T, F\}$

$f: CNF \longrightarrow 3CNF$

Replace each clause $(a_1 \vee \cdots \vee a_t)$ with

$$(a_1 \vee a_2 \vee x_1)(\bar{x}_1 \vee a_3 \vee x_2)(\bar{x}_2 \vee a_4 \vee x_3) \cdots\cdots (\bar{x}_{t-3} \vee a_{t-1} \vee a_t)$$

where $x_1, \dots, x_{t-3}$ are new variables

We need to check

1) f is Poly-time (easy)

2) φ is Sat iff $f(\varphi)$ is Sat

2) $(\Rightarrow)$ $T(\varphi) = \text{True}$ $\quad C = (a_1 \vee \cdots \vee a_t)$ clause of φ

Some i $\quad T(a_i) = \text{True}$

Set $T(x_j) = \text{True}$ for $j \le i-2$

$T(x_j) = \text{False}$ for $j > i-2$

$$(a_1 \vee a_2 \vee x_1)(\bar{x}_1 \vee a_3 \vee x_2) \cdots (\bar{x}_{i-2} \vee a_i \vee x_{i-1}) \cdots\cdots = C^*$$

2) ($\Leftarrow$) Suppose $T(C^*) = \text{True}$

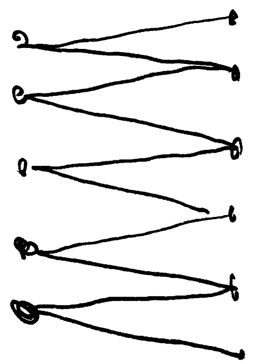
C^* has $t-2$ True clauses

<u>Claim</u> Some literal a_i s.t $T(a_i) = \text{True}$

<u>Matching Prob</u>

$t-3$ $\quad$ $t-2$

new variables $\quad x_1$



3-Clauses

(All True)

$\therefore$ Some 3-Clause all new literals are <u>False</u>

Thus Original literal must be true.

$$(a_1 \vee \cdots \vee a_t) \equiv C$$

<u>Def</u> A <u>clique</u> is a completely connected subgraph.

Clique $\equiv \{(G,k) \mid G$ is a graph with a k-clique $\}$

<u>Thm</u> Clique is NP-Complete

<u>Note</u> Clique $\in$ NP

Suffice to show: CNF $\leq_p$ Clique

ie f: cnf formula $\to$ (Graph, integer)

Suppose φ has clauses $\equiv C_1 \cdots C_K$

literals $\equiv a_1, \cdots, a_m$

<u>Def</u> $f(\varphi) = (G=(V,E), K)$

$V = \{ (a_i, C_j) \mid a_i \text{ is a literal in clause } C_j \}$

$E = \{ \langle (a,C), (a',C') \rangle \mid$ 1) $C \neq C'$ and 2) a is not negation of a' $\}$
$a \neq \bar{a}'$

<u>Claim</u> $\varphi \in CNF$ iff $f(\varphi) \in$ Clique

$\Rightarrow$ Suppose $T(\varphi) = \text{True}$

(At least one true literal per clause)

Say a_i is true in C_i

Then $\{ (a_1, C_1), \ldots, (a_k, C_k) \}$ is a clique in G.

$\Leftarrow$ Suppose H is a k-clique in G

u $H = \{ (a_1, C_1), \ldots, (a_k, C_k) \}$

1) C_i's are distinct

2) No $a_i = \bar{a}_j$

Set $T(x) = \begin{cases} \text{True} & \text{if } x = a_i \text{ some } i \\ \text{False} & \text{o.w.} \end{cases}$

<u>Def</u> $I \subseteq V$ is independent if $\forall x, y \in I \ (x, y) \notin E$.

The Independent Set Prob

$IND = \{(G, k) \mid G \text{ has an Ind set of size } k\}$

<u>Thm</u> IND is NP-Complete

$IND \in NP$

$\bar{G} = (V, \bar{E})$

$\bar{E} = \{(v, w) \in \breve{E} \mid (v, w) \notin E\}$

$f: (G, k) = (\bar{G}, k)$

Vertex Cover (VC)

Input: $G = (V, E)$; integer K

Question: $\exists C \subseteq V$ $|C| \leq K$

C contains at least one end-point of each edge

Thm: VC is NP-Complete

Will Show $IND \leq_p VC$

$f[(G,K)] = (G, n-K)$

Note $S \subseteq V$ is ind iff $V-S$ is a vertex cover.

Two Graph Problem

$G=(V,E)$ undirected (directed)

<u>Eulerian</u> cycle visits each edge exactly once.
<u>Hamiltonian</u> " " vertex " " "

<u>Def</u> G is <u>Eulerian</u> if G has an Eulerian cycle
G is <u>Hamiltonian</u> " " Ham " "

<u>Thm</u> G is Eulerian iff 1) G is connected
2) Every vertex has even degree

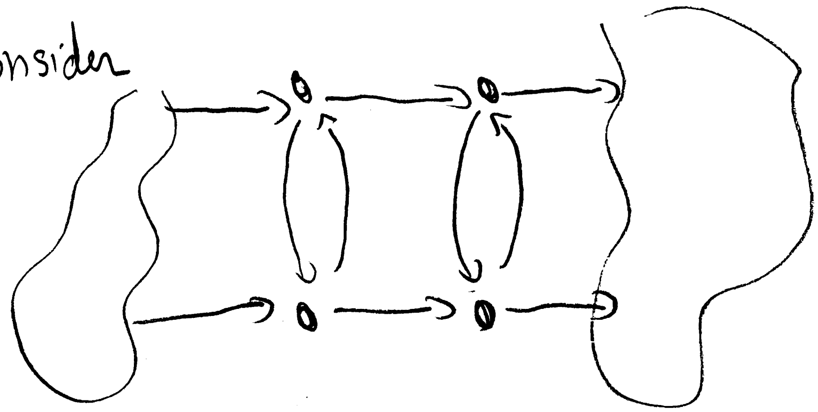
DHC ≡ Directed Hamiltonian Circuit Prob.

<u>Thm</u> DHC is NP-Complete

Will show Vertex Cover $\leq_p$ DHC

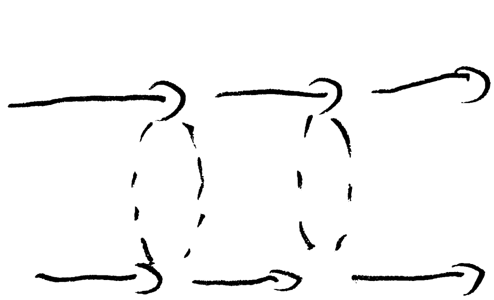
pf <u>Gadgets</u>

Consider



Only two "wirings"

straight through



Laced

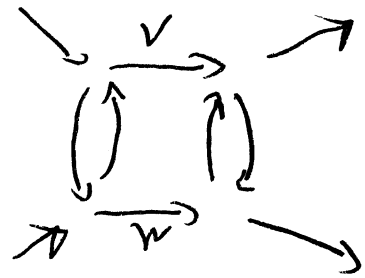
The construction of $G' = f(G,k)$ G is undir

G' is directed

<u>The</u> pieces of G'

for each $e = (v,w)$ of G

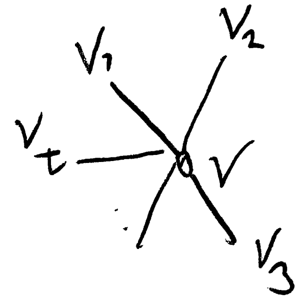
make

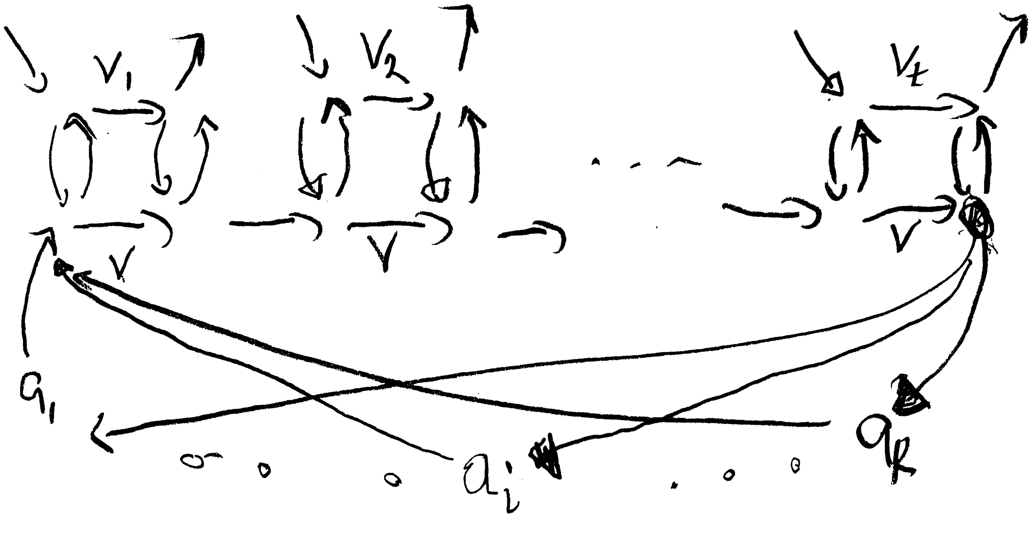


Plus vertices $a_1 \cdots a_k$ (one for each "cover vertex")

<u>Connecting the pieces:</u>

for each $v \in G$ say





Claim G has a K-cover iff G' is Hamiltonian

($\Rightarrow$) Suppose $\{u_1, \ldots, u_K\}$ is a vertex cover.

Use edges "$a_i \rightarrow u_i$".

If v also in cover then go straight thru

else Lace

note $\forall \{u_1, \ldots, u_k\} \subseteq V$ we get a simple cycle missing uncovered edges.

If $\{u_1, \ldots, u_K\}$ is a cover then it is Ham.

($\Leftarrow$) Similiar