

The Max Flow Prob

2/17/10

15-750

Input! 1) Flow network $G(V, E)$ directed (oriented)

2) Edge capacities $C: V \times V \rightarrow \mathbb{R}$

$$C(u, v) \geq 0$$

eg $(u, v) \notin E$ then $C(u, v) = 0$

3) $s, t \in V$ $s \equiv \text{source}$ & $t \equiv \text{sink}$

Goal: A flow $f: V \times V \rightarrow \mathbb{R}$ s.t.

1) Capacity constraints $f(u, v) \leq C(u, v)$

2) Skewed symmetric $f(u, v) = -f(v, u)$

3) Flow at a vertex $u \in V - \{s, t\}$

$$\sum_{v \in V} f(u, v) = 0 \text{ ie flow-in} = \text{flow-out}$$

$$\text{Net flow} \equiv |f| = \sum_{v \in V} f(s, v)$$

Max-Flow Prob:

input: Flow-Network $G = (V, E), s, t, c$

Output: Flow f with max net-flow

Do examples at end of lecture.

Residual Networks

Network $G=(V,E)$ & flow f

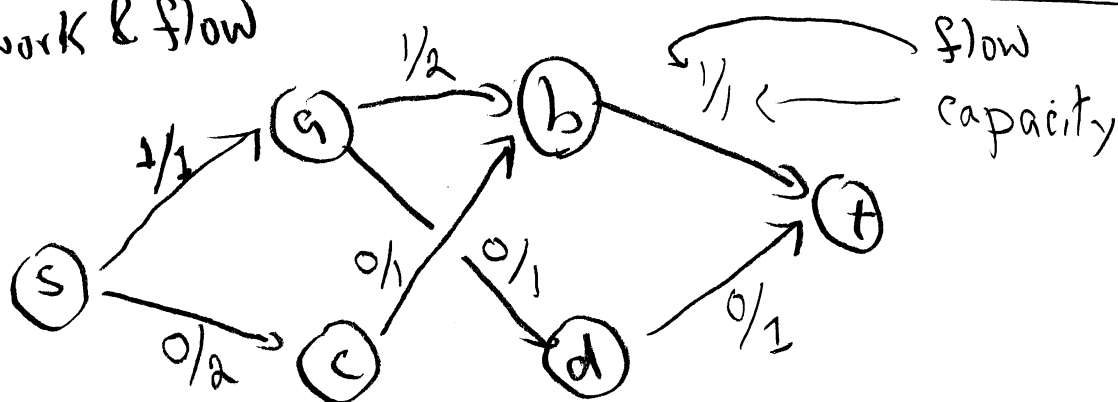
Residual Capacity: $C_f(u,v) = c(u,v) - f(u,v)$

Residual Network:

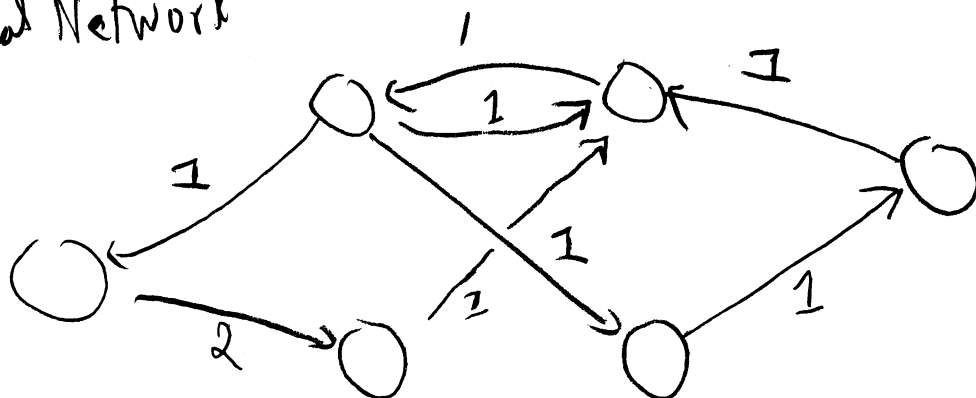
$$E_f = \{ (u,v) \in V^2 \mid C_f(u,v) > 0 \}$$

$$G_f(V, E_f)$$

Network & flow



Residual Network



Ford - Fulkerson

F-F Method (G, s, t, c)

- 1) Initialize flow f to 0
- 2) While $\exists$ aug path p in G_f
 Set $f = p + f$
- 3) return f

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Lemma f flow on G , G_f the residual

a) f' is flow on G_f iff $f+f'$ flow on G .

b) f' is max-flow on G_f iff $f+f'$ is max flow on G

c) $|f+f'| = |f| + |f'|$ (f' has no flow into s)

proof a) $f'(e) \leq c_f(e)$ iff $f'(e) \leq c(e) - f(e)$

iff $(f'+f)(e) \leq c(e)$

S, T cuts.

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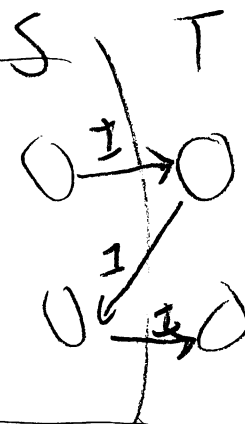
Def S, T is a cut if

$$1) S \cap T = \emptyset \text{ \& } S \cup T = V$$

$$2) s \in S \text{ \& } t \in T$$

$$Cap(S, T) \equiv \sum_{u \in S, v \in T} c(u, v)$$

eg

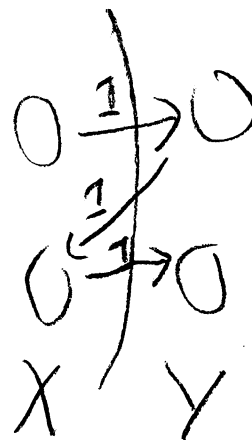


$Cap = 2$

f flow

$$f(X, Y) \equiv \sum_{u \in X} \sum_{v \in Y} f(u, v)$$

eg



$flow = 2$

$$f(S, T) \equiv \text{net flow from } S \text{ to } T$$

$$|f| \equiv f(s, V-s)$$

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Lemma If flow in G & (S, T) is a Cut
 then $|f| = f(S, T)$

proof induction on $|S|$

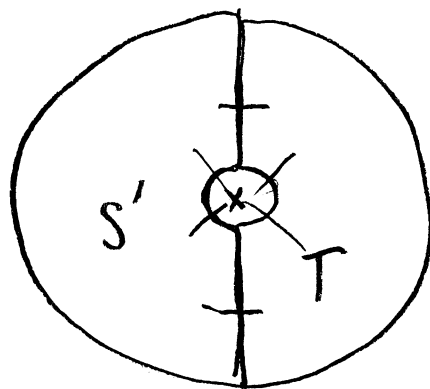
$|S|=1$ done

Assume true for $|S'| = k \geq 1$ cut (S', T')

$$S = S' \cup \{x\} \quad x \notin S' \quad x \neq T$$

$$T' = T' - \{x\}$$

$$\text{let } T = T \cup \{x\}$$



$$f(S, T) = f(S', T) + f(x, T)$$

$$f(S', T') = f(S', T) + f(S', x)$$

$$f(S, T) - f(S', T') = f(x, T) - f(S', x) = f(x, T) + f(x, S')$$

$$= f(x, V - \{x\}) = 0$$

$$\underline{\text{Cor}} \quad S, T \text{ cut} \quad |f| \leq c(S, T)$$

$$\text{since } f(S, T) \leq c(S, T)$$

Thm max-flow = min-cut
i.e. the following are equivalent

1) f is max-flow

2) G_f contains no augmenting path

3) $\exists$ cut S, T s.t. $|f| = \text{cap}(S, T)$

Proof $1) \Rightarrow 2) \Rightarrow 3) \Rightarrow 1)$

$$1) \Rightarrow 2) \text{ u } \neg 2) \Rightarrow \neg 1)$$

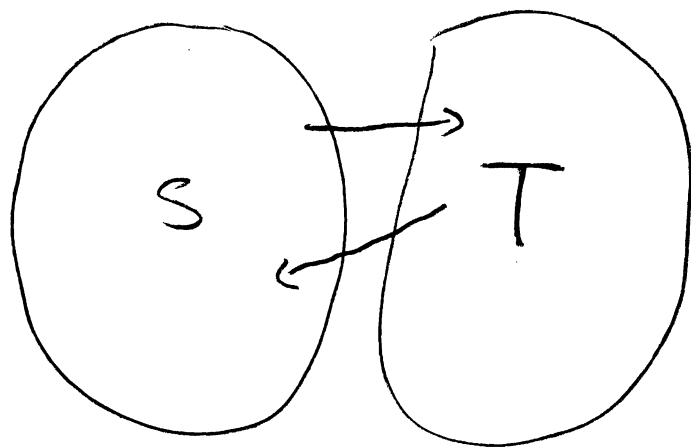
$\exists$ augmenting path $\Rightarrow f$ is not maximum

$$2) \Rightarrow 3)$$

Let $S \subseteq V \equiv$ reachable vertices from s
in G_f

$$T = V - S \text{ note: } t \in T$$

In G



1) f saturates all edges
from S to T

2) f does not use edges
from T to S

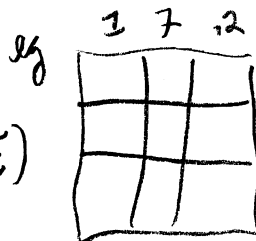
$$\therefore |f| = f(S, T) = \text{cap}(S, T)$$

3) $\Rightarrow$ 1) clear

An Image Segmentation Prob

Foreground / Background

Input: 1) Pixel Image



2) Affinity Graph. $G=(V,E)$

weights $P_{ij} \equiv$ similarity of pixel V_i & V_j

3) $b_j \equiv$ likelihood V_j in background

$a_j \equiv$ " foreground

Output: Partition A, B of V s.t.

$$\max_{A,B} g(A,B) = \sum_{i \in A} a_i + \sum_{i \in B} b_i - \sum_{\substack{i \in A \\ j \in B \\ (i,j) \in E}} P_{ij}$$

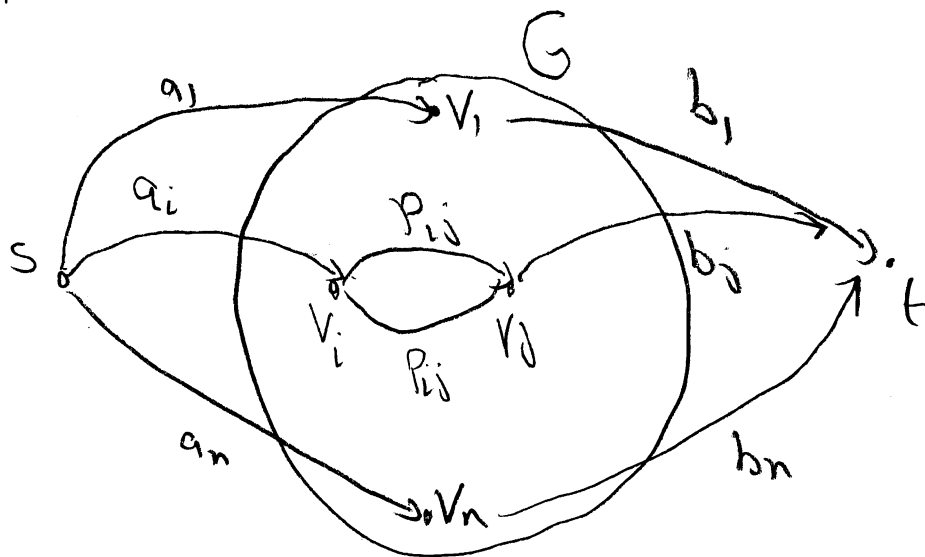
Change to min Prob.

$$Q = \sum_i (a_i + b_i) \text{ then}$$

$$g(A,B) = Q - \sum_{i \in A} b_i - \sum_{i \in B} a_i - \sum_{(i,j) \in E} P_{ij}$$

$$\text{Goal: Min } g'(A,B) = \sum_{i \in A} b_i + \sum_{i \in B} a_i + \sum_{(i,j) \in E} P_{ij}$$

Make a flow-Network:



Claim If A, B is a cut then

$$C(A, B) = g'(A, B).$$

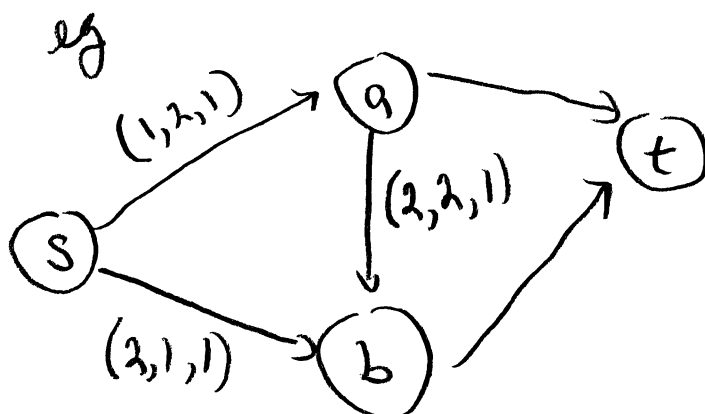
Thus Min Cut is a Max Segmentation.

Dynamic Networks

Input: Network: $G = (V, E)$

Discrete time: $0 \leq t \leq 2$

Capacities: $C_{ij}^t \equiv \text{cap from } a_i \text{ to } a_j \text{ at time } t.$



Buffer size: b_a^t

Idea: Make 4 copies of static network and add edges between copies.

