15-750

1/20/10

Binary Search Trees

Data Structure Dictionary

S is an ordered set.

1) Search$(K, S) \equiv K \in S?$

2) Insert$(K, S) \equiv$

3) Delete(K, S)

Note If 1), 2), 3) are the Design Requirement then use a Hash table

4) Range$(K, K', S) \equiv \{K'' \in S \mid K \leq K'' \leq K'\}$

If 1), ..., 4) use BST

Def A tree T is BST for keys S if:

1) T is an ordered binary tree with |S| nodes.
2) Each node stores a key.
3) Keys are in inorder

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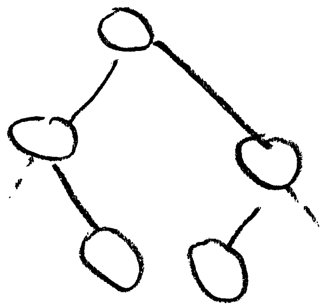
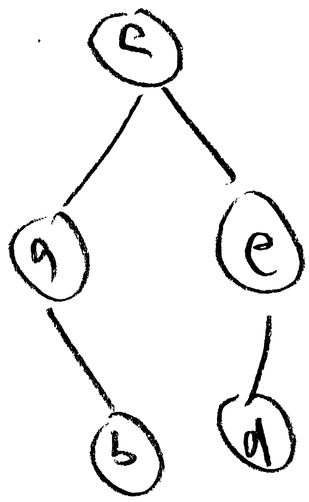
eg $S = \{a, b, c, d, e\}$

T =  $\Rightarrow$ BST 

T is <u>balanced</u> if max depth$(T) = O(\log n)$

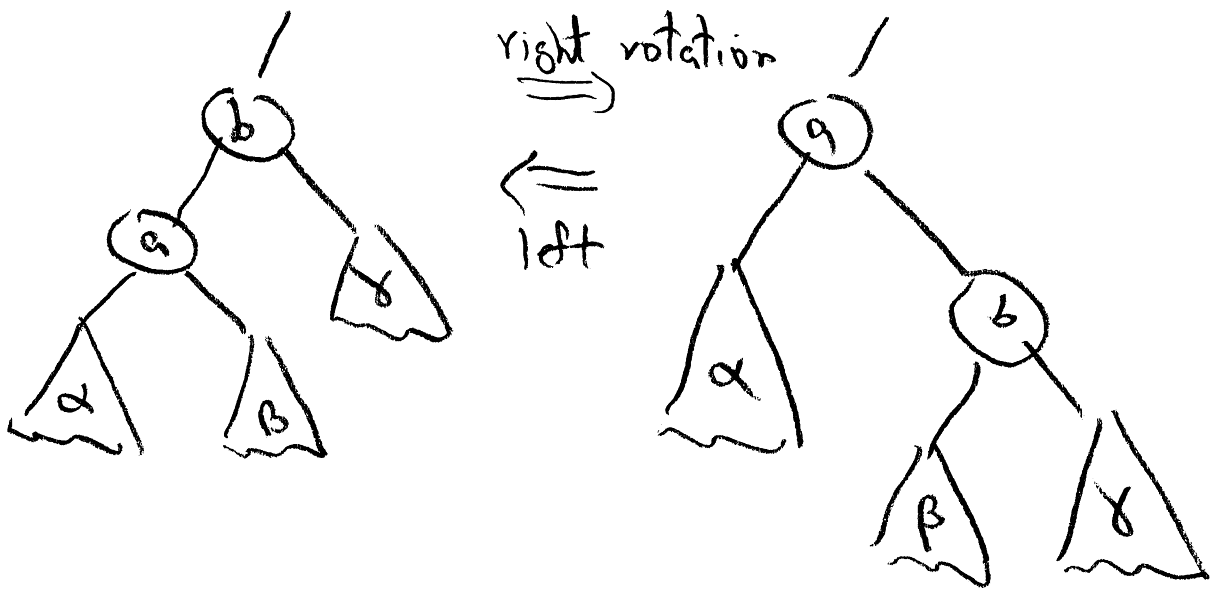
Types of Balanced BST's

Always Balanced - AVL, 2-3-4, RB, B-Trees

Randomized - skip-lists, Treaps ≡ tree-heaps

Amortized - Splay Trees

All these use the **Rotation**



To show: inorder is preserved

Applications

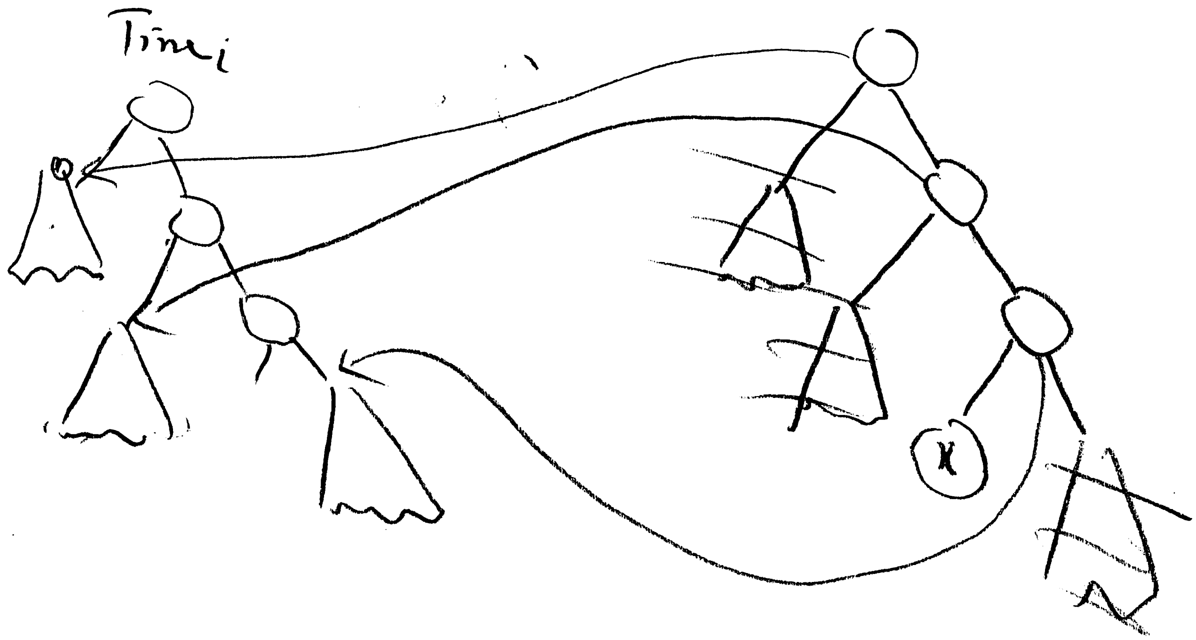
Persistance (undo)

First Idea Keep a tree for each time

Time

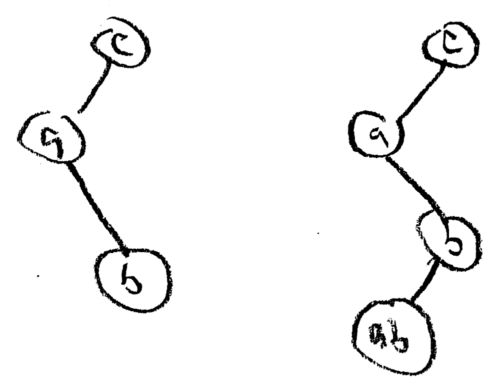
$O(n^2)$ spaces.

Idea 2 store only diffences Insert(x)



$O(\log n)$ space per new tree.

T VI(ab,T)



Vanilla-Insert (K T)

1) Find empty leaf node
2) add K to leaf

QS(A) Bad Quick Sort

1) Pick first a in A
2) Split S into $S<a$, a, $a<L$
3) Return QS(S), a, QS(L)

Eager

Vanilla-BST(A,T) VBST

Lazy

1) Extract first a from A
2) Return VBST(A, VI(a,T))

<u>Note</u> QS & V-BST do exactly the same comparisons but in different order.

Treaps

Keys $\equiv 1, \cdots, n$

Priorities $\equiv p(k)$ $\quad p(k) \neq p(l)$ for $k \neq l$

$T \equiv$ tree with a key at each node.

Def T is in heap order if $\forall x \in T$ x not root

$$p(\text{parent}(x)) < p(x)$$

Lemma A heap order BST exists and is unique.

pf Vanilla-insert in priority order.

Random Treap

Insert (K) ≡ 1) insert K into a leaf
2) pick random p(K)
3) rotate K up until in heap order

Delete (K) ≡ 1) Rotate K to a leaf
by picking highest priority child
2) remove K.

Correctness:

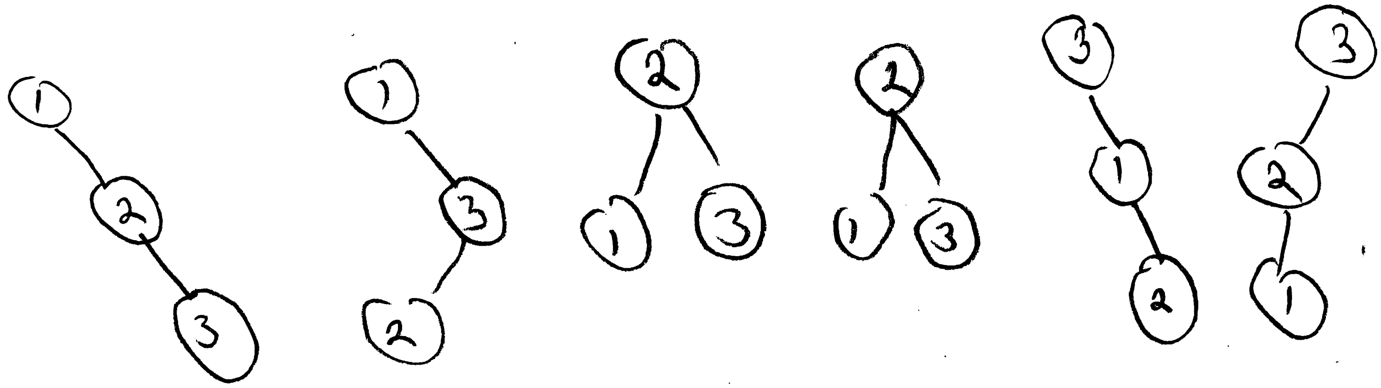
Expected Cost for Treaps

Goal: Determine expected # of comparisons to search(m.5, K) $\equiv S(m.5)$

eg $K = \{1,2,3\}$ & search(2.5, K)

Treaps insert order

(123) (132) (213) (231) (312) (321)



	(123)	(132)	(213)	(231)	(312)	(321)
#	3	3	2	2	3	2

$$S(2.5) = \frac{15}{6} = 2\tfrac{1}{2}$$

Note for random BST $\frac{13}{5} = 2\tfrac{3}{5}$

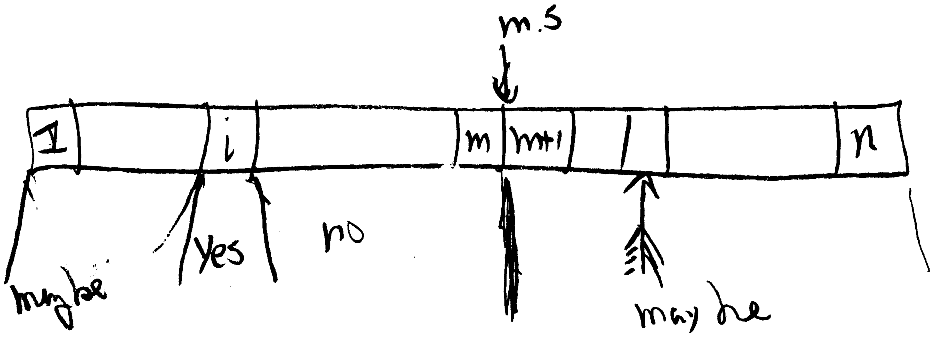
Keys $\{1, \ldots, n\}$ in Treap

$C(i,m)$ = Event [i is compared to m in Search(m)]

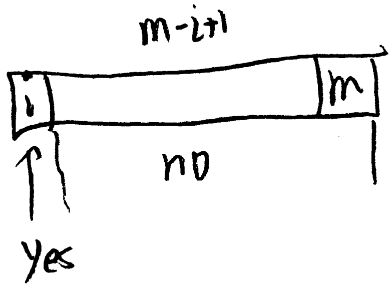
<u>Claim</u> $\text{Prob}[C(i, m.5)] = \frac{1}{m-i+1}$ if $i \le m$

$= \frac{1}{i-m}$ if $i > m$

Treap construction as dart game



informal argument: WLOG board is



$\text{Prob}[C(i, m.5)] = \frac{1}{m-i+1}$

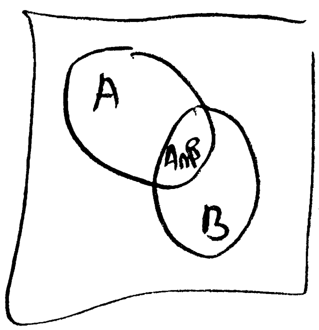
Formal Arugment

B_j = Event [jth dart is first to land in $[i, \ldots, m]$]

$B_k \cap B_j = \emptyset$ if $k \neq j$ $\quad \Pr\left(\bigcup_{j=1}^{\infty} B_j\right) = 1 \qquad (*)$

Conditional Probability: Let A, B be events

$$\Pr[A|B] = \frac{\Pr[A \cup B]}{\Pr[B]}$$



$i \leq m$

$$\Pr[\underbrace{C(i,m,5)}_{=A}] = \sum_{j=1}^{\infty} \Pr[B_j] \Pr[A|B_j] \quad \text{true by } (*)$$

note $\Pr[A|B_j] = \frac{1}{m-i+1}$

$$\Pr[A] = \sum_{j} \Pr[B_j]\left(\frac{1}{m-i+1}\right) = \frac{1}{m-i+1} \sum_{j=1}^{\infty} \Pr[B_j] = \frac{1}{m-i+1}$$

QED

$S(m.5)$ = # comparison search m.5

$$= \sum_{i \neq m} C(i, m.5)$$

$$E(S(m.5)) = \sum E(C(i, m.5)) = \{\text{Prob}[C(i, m.5)]\}$$

$$= \sum_{i \leq m} \frac{1}{m-i+1} + \sum_{i > m}^{n} \frac{1}{i-m} \leq 2\sum_{i=1}^{n} \frac{1}{i} = 2H_n$$

$$= 2\ln n + O(1)$$

Expect # of Comparisons for

Insert, Search, Delete $= O(\log n)$

Counting Rotations

2 Cases: Insert & Delete

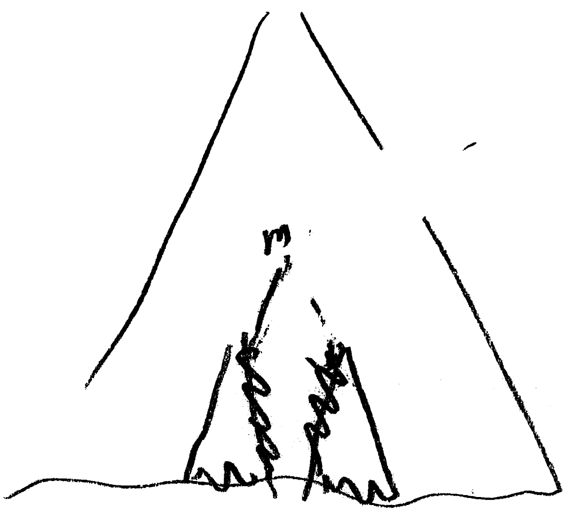
Delete : Move-to-Leaf

Thm : Expect #rotations < 2

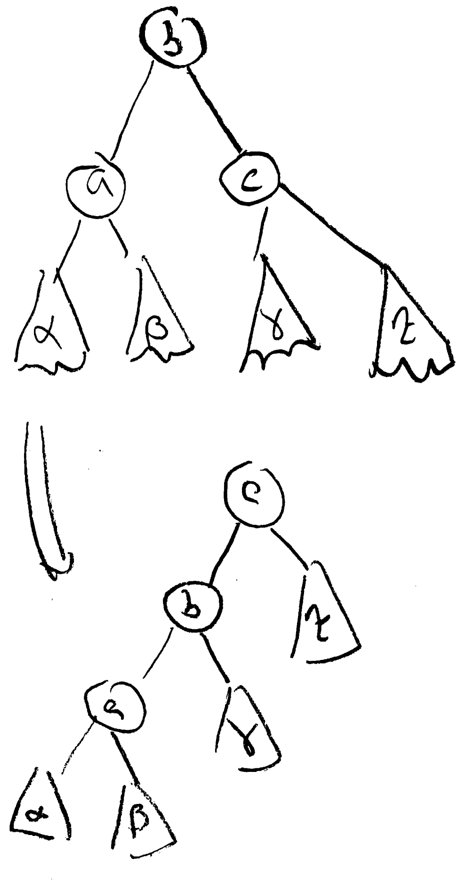
Claim Possible pivots with m are

1) Right-most nodes in left subtree of m
2) Left " " right " "

eg



pf



Def

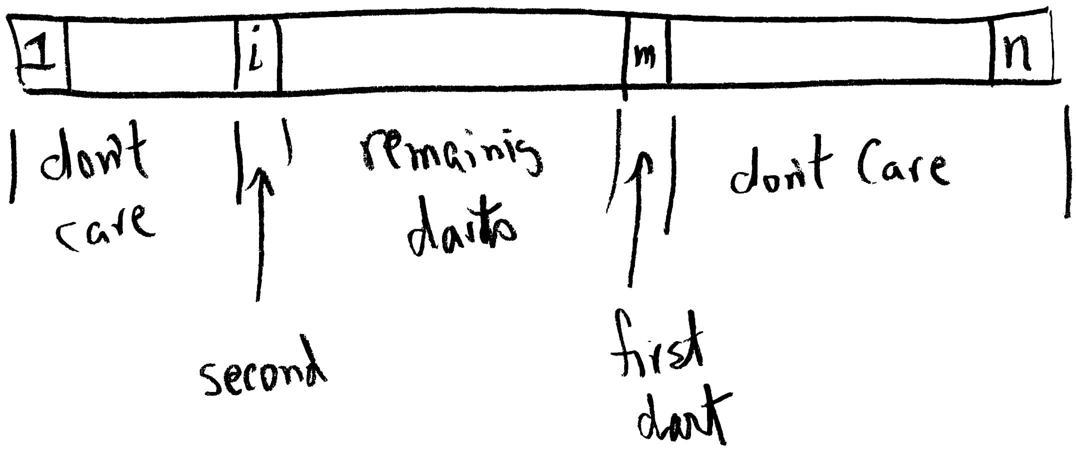
D_i = Event[i is a right most node in left subtree of m]

D'_i = Event[i " "left" " "right" "]

Claim $\Pr[D_i] = \left(\frac{1}{m-i+1}\right)\left(\frac{1}{m-i}\right)$

pf Dart Game (Informal)

Needed so that D_i:



$$\Pr(D_i) = \left(\frac{1}{m-i+1}\right)\left(\frac{1}{m-i}\right)$$

$$\Pr(D'_i) = \left(\frac{1}{i-m+1}\right)\left(\frac{1}{i-m}\right)$$

<u>Def</u> R_m = # rotations in move-to-leaf of m

$$E(R_m) = \sum_{i<m} E(D_i) + \sum_{i>m} E(D_i')$$

$$\le \sum \frac{1}{(m-i+1)(m-i)} + \sum \frac{1}{(i-m+1)(i-m)}$$

$$\le \sum_{i=1}^{m-1} \frac{1}{(i+1)\,i} + \sum_{i=1}^{n-m} \frac{1}{(i+1)\,i}$$

<u>note</u> $\frac{1}{(i+1)\,i} = \frac{1}{i} - \frac{1}{i+1}$

thus $= \sum_{i=1}^{m-1} \frac{1}{i} - \sum_{i=2}^{m} \frac{1}{i} = 1 - \frac{1}{m} < 1$

$$E(R_m) < 2$$