Approximation Algorithms

15-750
4/26/10
CLRS Chap 35

Goal: Solve NP-Hard Problems Approximately

We now consider optimization problems

e.g. 1) Coloring a graph with min # of colors
2) Finding a min size vertex cover

Let P be an optimization prob.

Def An alg A is poly-time K-approx alg if.
1) A is poly-time
2) $A(\tau) \leq K \, Opt_P(\tau)$

Vertex-Cover Prob

Input: $G = (V, E)$

Output: Minimum size $C \subseteq V$ that covers E

Question: How do we lower bd Opt?

One Answer for VC:

<u>Def</u> $e, e' \in E$ are <u>independent</u> if $e \cap e' = \emptyset$

Let $E' \subseteq E$ be an ind set of edges.

<u>Claim</u> $Opt \geq |E'|$

<u>pf</u> Opt must contain at least one endpoint from each $e \in E'$.

Alg Approx-VC

1) Find a <u>maximal</u> ind set of edges E'

2) Return V' the endpoints of E'.

<u>Note</u> Approx-VC returns a vertex-cover.

<u>Claim</u> Approx-VC is a poly-time 2-approx alg.

pf 1) Clearly Approx-VC is poly-time

2) $|\text{Approx-VC}| \leq 2|E'| \leq 2|\text{Opt-VC}|$

$2-\varepsilon$-approx thought hard.

Traveling-Salesman Prob

Input: Complete graph $G=(V,E)$

Cost fcn $C: E \rightarrow R^+$

Output: Min-cost Hamiltonian cycle.

Claim TSP is NP-Hard

pf Ham-Cycle $\leq_p^T$ TSP

$$G=(V,E) \Rightarrow C(u,v)=\begin{cases} 0 & \text{if } (u,v)\in E \\ 1 & \text{o.w.} \end{cases}$$

$G \in$ Ham-Cycle iff $|TSP(C)|=0$

TSP with Triangle Inequality

Def c satisfies the triangle inequality if

$$\forall u,v,w \in V \quad c(u,w) \le c(u,v) + c(v,w)$$

Claim TSP with tri-inequality is NP-Hard.

Pf Ham-Cycle $\le_p^T$ TSP with TI

$$G=(V,E) \qquad c((u,v)) = dist(u,v)$$

$$\text{Opt-TSP}(G,c) = n \text{ iff } G \in \text{Ham-Cycle}$$

Question: Lower Bound Opt-TSP?

Let T^* be Opt Tour.

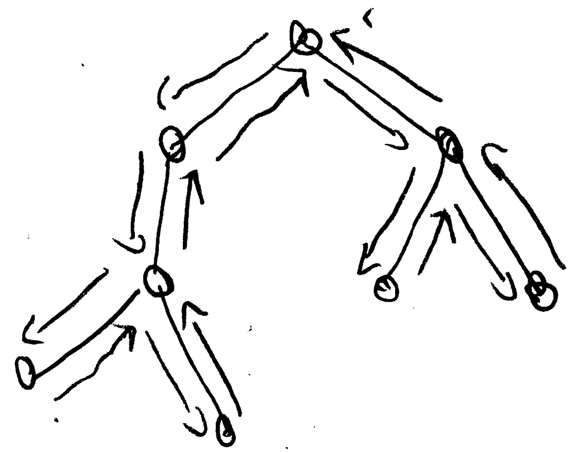
Thus T^* minus some edge is a spanning Tree.

$\therefore |MST| \le |T^*|$

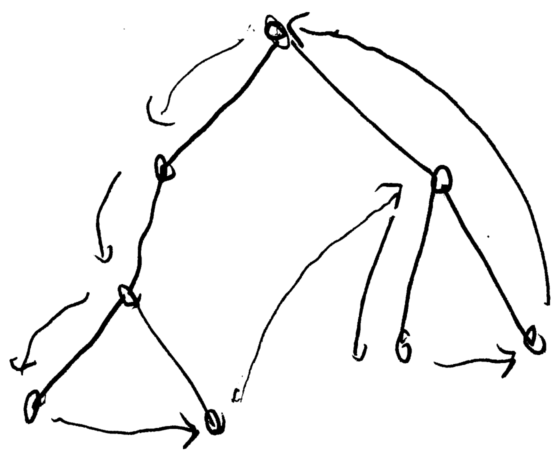
Approx-TSP Alg

1) Find a MST T of G
2) Compute an Euler Tour of T.
3) Remove multiply visited vertices from Euler Tour.

EG



Euler Tour



Ham-Cycle

Let Tour be an Euler-Tour with extra vertices shunted.

$$|\text{Tour}| \underset{\Delta}{\leq} |ET| \leq 2|MST| \leq 2 \cdot \text{Opt-TSP}$$

<u>Thm</u> If $P \neq NP$ then $\forall \rho > 1 \nexists$ poly-time ρ-approx for TSP.

<u>pf</u>

To Show: Ham-Cycle $\leq_p^T$ ρ-approx TSP

Input: $G=(V,E)$

Output: $G'=(V,E')$ $E'=\{(u,v) \mid u \neq v\}$

$$C(u,v) = \begin{cases} 1 & \text{if } (u,v) \in E \\ \rho \cdot n+1 & \text{o.w} \end{cases} \qquad n = |V|$$

<u>Note</u> $G \in$ Ham-Cycle then $|TSP| = n$

$G \notin$ Ham-Cycle then $|TSP| \geq (n-1) + \rho n + 1$

$$= (\rho+1)n$$

<u>Thus</u> If $G \in$ Ham-Cycle & Appox-TSP returns a ρ-approx it must be a Ham-Cycle.

Set-Cover Prob.

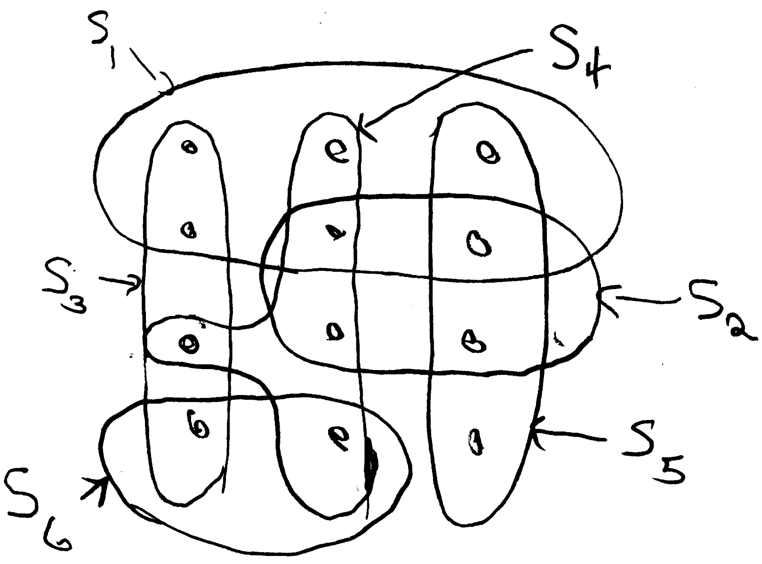
Input: Finite set X; Collection of subset $\mathcal{S}$

Output: Min size cover.

Greedy-Set Cover $(X, \mathcal{S})$

1) $\mathcal{U} \Leftarrow X$
2) $\mathcal{C} \leftarrow \emptyset$
3) <u>While</u> $\mathcal{U} \neq \emptyset$ <u>do</u>
   a) Select $S \in \mathcal{S}$ with $\max(S \cap \mathcal{U})$
   b) $\mathcal{U} \Leftarrow \mathcal{U} - S$
      $\mathcal{C} \leftarrow \mathcal{C} \cup \{S\}$
4) Return $\mathcal{C}$

EG



Greedy: S_1, S_4, S_5, S_3 or S_6

Opt: S_3, S_4, S_5

Def $H(d) = \sum_{i=1}^{d} 1/i$

Thm Greedy-Set Cover is Poly-time ρ-approx for set-cover where

$\rho = H(d)$ $d = \max\{|S| : S \in \mathcal{S}\}$

A Worst Case Example

$X = \{x_1, x_1', \ldots, x_n, x_n'\}$

$S = \{x_1, \ldots, x_n\}$

$S' = \{x_1', \ldots, x_n'\}$

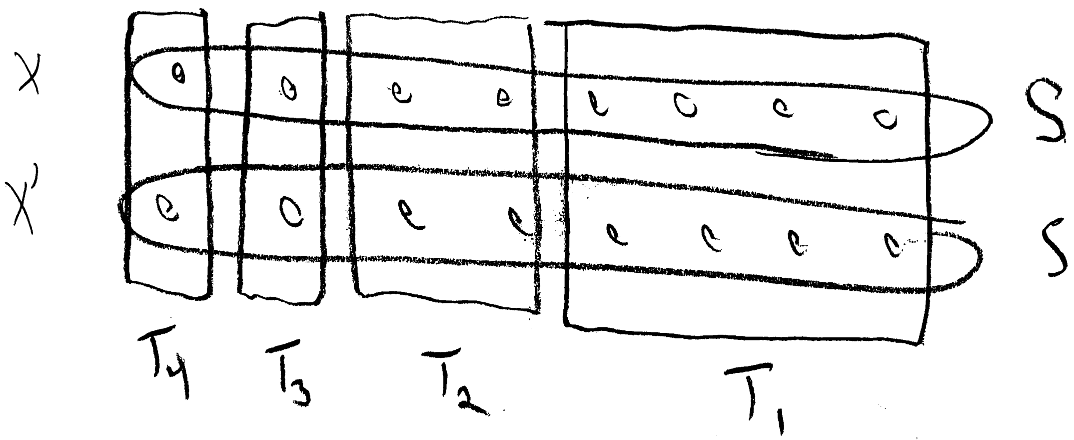
$T_1 = \{X_{n/2}, X_{n/2}', \ldots, X_n, X_n'\}$

$\vdots$

$T_i = \{X_{n/2^i}, X_{n/2^i}', \ldots, X_{n/2^{i-1}}, X_{n/2^{i-1}}'\}$

$\vdots$

$T_{\log} = \{X_1, X_1'\}$



$OPT = \{S, S'\}$

$Greedy = \{T_1, \ldots T_{\log n}\}$

pf Let $C \subseteq \mathcal{S}$ sets picked by Greedy

$C^* \subseteq \mathcal{S}$ an Opt set

$C = \langle S_1, \ldots, S_k \rangle$ set in order of selection.

Def $x \in X$

$$C_x = \frac{1}{|S_i - (S_1 \cup \cdots \cup S_{i-1})|}$$ where i = first i s.t. $x \in S_i$

Note Denomiator $\equiv$ # elements covered at step i.

$\therefore$ $k = |C| = \sum_{x \in X} C_x$

Claim $\sum_{x \in S} C_x \leq H(|S|)$

pf later

pf using claim

$$|\mathcal{C}| = \sum_{x \in X} c_x \leq \sum_{S \in \mathcal{C}^*} \sum_{x \in S} c_x \quad (\mathcal{C}^* \text{ is a cover})$$

$$\leq \sum_{S \in \mathcal{C}^*} H(|S|) \quad \text{(by claim)}$$

$$\leq |\mathcal{C}^*| \, H(\max\{|S| : S \in \mathcal{S}\})$$

pf of Claim $\sum_{x \in S} C_x \leq H(|S|)$

fix $S \in \mathcal{S}$

Def $u_i = |S - (S_1 \cup \cdots \cup S_i)|$

note $|S| = u_0 \geq u_1 \geq \cdots \geq u_k = 0$

Let $t = \min_t \{u_t = 0\}$

note $u_{i-1} - u_i$ = # of element in S covered at time i.

We def the cost to cover one such element

$$C_x = \frac{1}{|S_i - (S_1 \cup \cdots S_{i-1})|}$$

Since Greedy is greedy S_i covers at least as many as S

Thus $|S_i - (S_1 \cdots S_{i-1})| \geq |S - (S_1 \cup \cdots \cup S_{i-1})| = u_{i-1}$

$$\therefore \sum_{x \in S} c_x \leq \sum_{i=1}^{t} (u_{i-1} - u_i) \frac{1}{u_{i-1}} \qquad (*)$$

note $(u_{i-1} - u_i) \frac{1}{u_{i-1}} \leq \sum_{j=u_i+1}^{u_{i-1}} 1/j$

thus $(*) \leq \sum_{j=1}^{|S|} 1/j = H(|S|)$

QED