

The Preflow-Push Max-Flow Alg

15-750
2/16/09

Goal: $O(n^3)$ Alg

This Alg + Better Data Structures: $O(n \cdot m \log n)$

Known: $O(m \sqrt{n} \log^k n)$ (some k)

Idea Relax the flow-in = flow-out condition.

Def f is an s-t preflow (preflow) if

- 1) $f: E \rightarrow \mathbb{R}^+$
- 2) $0 \leq f(e) \leq c_e$
- 3) $e_f(v) = \sum_u f(u, v) - \sum_u f(v, u) \geq 0$

As before we have a residual graph.

G_f

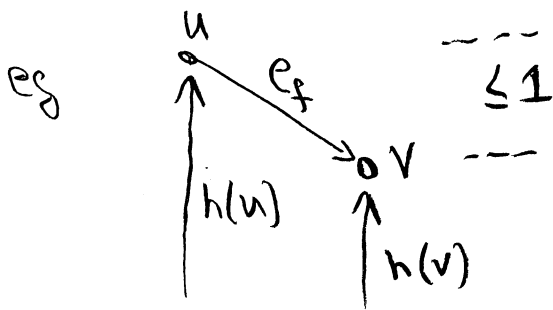
We use a vertex label to guide "pushing" the flow.

Let $h: V \rightarrow \mathbb{Z}^+ = \{0, 1, 2, \dots\}$ (label or height)

Def f & h are compatible if

1) $h(t) = 0$ & $h(s) = n$ (sink & source condition)

2) $\forall (u, v) \in E_f \quad h(v) \leq h(u) + 1$



Lemma f (st preflow) & h (compatible labeling)

then no augmenting path in G_f .

pf By contradiction! Let P be such a path

We may assume P is simple. # edges of $P < n$

∴ $h(s) - h(t) < n$ contra!

Cor f (s-t flow) & h (compatible labeling)
 then f is a Max-Flow.

$$\text{Init}(G) \quad f(s, v) = C(s, v) \quad \forall v$$

$$f(e) = 0 \quad \text{o.w.}$$

$$h(s) = n \quad h(v) = 0 \quad \text{o.w.}$$

Note f & h are compatible

Pt Only v st $h(v) > 0$ is s .

But we saturated edges out of s .

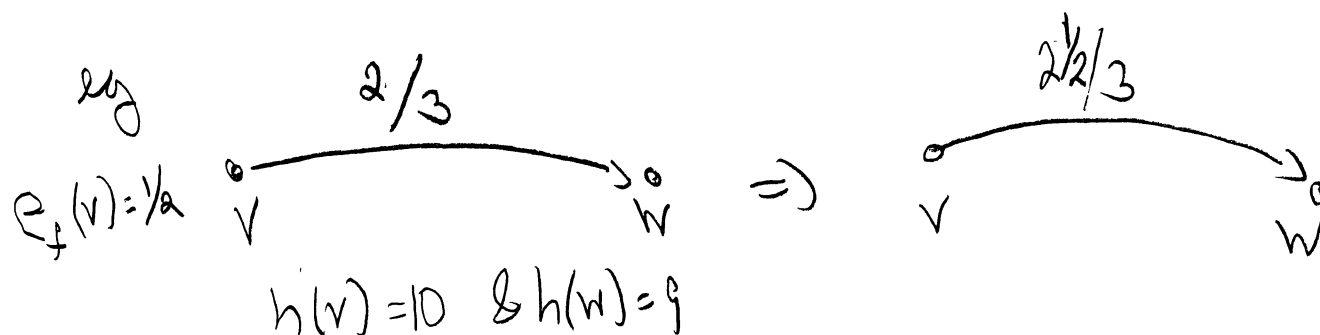
Push & Relabel

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Push (f, h, v, w)

if $e_f(v) > 0$ & $(v, w) \in E_f$ & $h(v) > h(w)$

then "push" $\min\{e_f(v), c_f\}$



Relabel (f, h, v)

If $e_f(v) > 0$ & $\forall (v, w) \in E_f$ $h(v) \leq h(w)$

then $h(v) \leftarrow h(v) + 1$

Preflow-Push

1) Init

2) While $\exists v \neq t$ st $e_f(v) > 0$ do

If $\exists w$ st $(v, w) \in E_f$ & $h(v) > h(w)$

then $\text{Push}(f, h, v, w)$

Else $\text{relabel}(f, h, v)$

3) Return f

Correctness

Claim 1 f & h are always compatible.

Claim 2 If Preflow-Push returns f then
 f is a max-flow

pf of claim 1

Each push preserves compatibility.

" " relabel "

" "

Timing & Termination

To show excess can be "pushed" back to s .

Lemma f preflow & $e_f(v) > 0$ then $\exists$ path from v to s . (s not t)

pf Let $A = \{w \mid \exists \text{ path from } w \text{ to } s\}$
 $B = V - A$

Note Flow can only go from B to A .

Claim $\sum_{v \in B} e_f(v) = 0$

$$\sum_{v \in B} e_f(v) = \sum_{v \in B} f^{\text{in}}(v) - f^{\text{out}}(v)$$

Three types of edges $(u, v) \in \text{Sum}$

- 1) $u, v \in B$ (cancel)
- 2) $u \in A$ & $v \in B$ (zero)
- 3) $v \in B$ $u \in A$ (negative in sum)

is $0 \leq \sum_{v \in B} p_f(v) = -f^{\text{out}}(B) \leq 0$ QED

Claim $h(v) \leq 2n-1$

$$L = \frac{v}{h(r)}$$
 $n-1$

$s \quad h(s) = n$

$$\Rightarrow h(v) \geq 2h-1$$

Cor # of relabels $< 2n^2$

Count # of pushes

2 types

saturating

$$p_{\text{push}} \in C_f$$

nonsaturating

Otherwise

Claim # saturating pushes $\leq 2nm$

to show # per edges $\leq n$

Before push $v \xrightarrow{p_f} w$ $h(v) = h(w) + 1$

After push $v \xleftarrow{e_f} w$ need 2 relabels.

Lemma # nonsaturating pushes $\leq 4n^2m$.

pt Consider the following potential

$$\Phi(f, h) = \sum_{e_f(v) > 0} h(v)$$

ops	#	AC	total	unit-cost
relabel	$2n^2$	≤ 2	$4n^2$	1
saturating	$2nm$	$2n-1$	$2nm(2n-1)$	1
non-sat	?	0	0	1

$\Rightarrow ? \leq 4n^2m$

We have shown

Thm Preflow-Push is $O(n^2 m)$ alg.

New alg Preflow-Push-Max-Height

$$\text{Let } H = \max_{e_f(v) > 0} h(v)$$

Alg Push(v) st $e_f(v) > 0$ & $h(v) = H$

Thm Preflow-Push-Max-Height in $O(n^3)$

$$\Phi(f, h) \approx H$$

Note: Only relabel increases Φ .

$2n^2$ increases and $4n^2$ total changes.

Goal: # of non-sat pushes per change.

at most one per node or n .

of non-sat pushes is $\leq 4n^3$.