Resistive Model of a Graph & Random Walks

Motivation: Making a recommendation (NETFLIX)

Viewers

Movies

ranking

Question: Should we recommend M to V?

Score(V,M)

Idea 1

Score(V,M) = graph dist from V to M

\[ W_{ij} = \frac{1}{\text{rank}_{ij}} \]

\[ \text{Score}(V,M) = \min_{VPM} W(P) \]

Idea 2

\[ W(P) = \min_{e \in P} (\text{rank}(e)) \]

\[ \text{Score}(V,M) = \max_{VPM} W(P) \]
Problem: For 1) and 2) extra paths do not improve score.

Idea 3: \( \text{Score}(V, M) = \text{Max flow from } V \text{ to } M \).

Problem: Shorter path do not improve score.

Idea: View edges as conductors.
\( \text{Score}(V, M) = \text{effective conductance} \).
Resistance Theory

Ohms Law:

\[ \frac{V}{R} = \frac{1}{C} \]

\[ V = \text{voltage} \]
\[ R = \text{resistance} \]
\[ i = \text{current} \]
\[ C = \frac{1}{RC} \]

\[ i = C \cdot V = \frac{V}{R} \]

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Facts no proof

Resistors in series

\[ R = R_1 + \cdots + R_m \]

\[ C = \frac{1}{(1/C_1 + \cdots + 1/C_m)} \]

i.e. \[ i = \frac{V}{R} \]
Conductors in Parallel

\[ C = C_1 + \ldots + C_m \]
\[ i = V \cdot C \]

**Effective Resistance/Conductance**

Let \( G \) be a network of resistors

\[ i_{ab} = \text{current} \]
\[ V_{ab} = \text{voltage} \]

\[ \text{Def: } R_{ab} = \frac{V_{ab}}{i_{ab}}, \quad \text{and } C_{ab} = \frac{1}{R_{ab}} \]
HW) Show that $R_{ab}$ is a metric space

ie

1) $R_{ab} \geq 0$
2) $R_{ab} = 0$ iff $a = b$
3) $R_{ab} = R_{ba}$
4) $R_{ac} \leq R_{ab} + R_{bc}$
Computing effective resistance

Use Kirchhoff's Law in flowin = flowout

An example

\[ V_1 \rightarrow C_1 \rightarrow V \rightarrow C_2 \rightarrow V_2 \]

by Ohm's Law

\[ i_1 = C_1 (V - V_1) \]
\[ i_2 = C_2 (V - V_2) \]
\[ i_3 = C_3 (V - V_3) \]

Residual current \( i_1 + i_2 + i_3 \)

by Kirchhoff

\[ i_1 + i_2 + i_3 = 0 \]

\[ C_1 (V - V_1) + C_2 (V - V_2) + C_3 (V - V_3) = 0 \]

\[ (C_1 + C_2 + C_3) V = C_1 V_1 + C_2 V_2 + C_3 V_3 \]
\[ C = C_1 + C_2 + C_3 \]
\[ CV = C_1 V_1 + C_2 V_2 + C_3 V_3 \]
\[ V = \frac{C_1}{C} V_1 + \frac{C_2}{C} V_2 + \frac{C_3}{C} V_3 \]

\( V \) is convex combination of \( V_1, V_2, V_3 \)

\[ \text{residual current} = CV - C_1 V_1 - C_2 V_2 - C_3 V_3 \]

The general case
\[ G = (V, E, C) \quad C: V \to \mathbb{R}^+ \]
\[ V = \{ V_1, \ldots, V_n \} \]

\[ d(V_i) = \sum_{(i,j) \in E} C_{ij} \]

\[ A_{ii} = \begin{cases} C_{ii} & \text{if} \ (i,i) \in E \\ 0 & \text{otherwise} \end{cases} \]
\[ \text{Laplacian}(G) = L(G) = L \]

\[ L_{ij} = \begin{cases} 
    d(v_i) & \text{if } i = j \\
    -w_{ii} & \text{if } (i,j) \in E \\
    0 & \text{otherwise}
\end{cases} \]

\[
L = D - A \quad \text{where } D = \begin{pmatrix} 
    d(v_1) & 0 \\
    0 & \cdots & 0 \\
    0 & \cdots & d(v_n)
\end{pmatrix}
\]

Let \( V \) be a voltage setting of nodes.

\( (LV)_i \) = residual current at \( V_i \)

Inverse: We inject current and get voltages.

The net injected must be zero!
Goal: \( R_{in} \)

method 1 solve
\[
\begin{pmatrix}
0 \\
\vdots \\
V_{n-1} \\
V_n \\
1
\end{pmatrix}
\begin{pmatrix}
i \\
\vdots \\
i \\
\vdots \\
i
\end{pmatrix}
\begin{pmatrix}
0 \\
\vdots \\
0 \\
0 \\
i
\end{pmatrix}
\]

\[ i = \frac{V}{R} \]
\[ i=1 \] then \( R = V \)

return \( \frac{1}{i} \)

\((*)\) is called a boundary valued prob.

In our case \( V_1 \& V_n \) are the bdary

\( (V_1, \ldots, V_n) \) is called harmonic

because \( V_i \) e interior \( \Rightarrow \)

\( V_i \) is convex combination of neighbors
Maximum Principle: If $f$ is harmonic then min & max are on bdary.

If $v_i$ & $v_j$ at $v_i \leq v \leq v_j$.

Uniqueness Principle: If $f$ & $g$ are harmonic with same bdary values then $f = g$.

If $f - g$ is harmonic with zero on bdary.

$\Rightarrow f - g = 0 \Rightarrow f = g$.
Method 2: solve $LV = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$ Does $V$ exist?

$R_{in} = V_i - V_n$

Another way to view the Laplacian

**Edge-Vertex Matrix**

\[ \Gamma = \begin{pmatrix}
  & V_i & -V_n \\
  e_1 & 1 & -1 & 0 & 0 \\
  e_2 & 0 & 1 & 0 & 1 \\
  e_3 & 0 & 1 & -1 & 0 \\
\end{pmatrix} \]

Orient each edge
Let $C_1, \ldots, C_m$ = conductance of $e_1, \ldots, e_m$

$$C = \begin{pmatrix} C_1 & 0 \\ 0 & C_m \end{pmatrix}$$

Note

$\Gamma V$ = voltage drop across each edge

$\Gamma^T \Gamma V$ = current flow

$\Gamma^T \Gamma V$ = residual current at each vertex

Thus

$L = \Gamma^T \Gamma$
Current & Energy / Power Dissipation

\[ \frac{C}{R} \]

[Diagram of a circuit with a voltage source and a current label]

Newton Energy = Force \cdot Distance
\[ \equiv \text{Volt} \cdot \text{Current} \]
\[ \equiv V \cdot I \]
\[ \equiv C \cdot V^2 \]
\[ \equiv I^2 R \]

Network
\[ E = \frac{1}{2} \sum_{x,y} i_{xy} (V_x - V_y) \]

\[ V^T L V = V^T \Gamma^T C \Gamma X = (\Gamma X)^T C \Gamma X \]
\[ = \sum_{(x,y) \text{ oriented}} (V_x - V_y)^2 = E \]
Suppose \( a, b \text{ eV effective resistance } R_{ab} \)

Effective energy \( i_{ab}^2 R_{ab} = R_{ab} \text{ if } i_{ab} = 1 \)

Real energy using Kirchhoff's Law

Solve \( LV = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \quad V_a = V_i \quad V_b = V_n \)

Energy \( = V^T L V = V^T \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = V_a - V_b \)

The real Kirchhoff energy = effective energy

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Minimum Energy flow

Def (old definition)

\( j : E \rightarrow R \) is a flow from \( a \) to \( b \)

\( j_{xy} = -j_{yx} \)

\( \sum_y j_{xy} = 0 \text{ if } x \neq a, b \)

\( j_{xy} = 0 \quad (x, y) \notin E \)
Def: \( \dot{j}_x = \sum_{y} \dot{j}_{xy} \) (residual flow at \( x \))

Let \( W \) = any voltage settings
\( \dot{j} = \) any flow from \( a \) to \( b \)

Conservation of Energy
\[
(W_a - W_b) \dot{j}_a = \frac{1}{2} \sum (W_x - W_y) \dot{j}_{xy}
\]

\[
\sum (W_x - W_y) \dot{j}_{xy} = \sum_{x} W_x \sum_{y} \dot{j}_{xy} - \sum_{y} W_y \sum_{x} \dot{j}_{xy}
\]

\[
= W_a \sum_{y} \dot{j}_{ay} + W_b \sum_{y} \dot{j}_{by} - (W_a \sum_{x} \dot{j}_{xa} + W_b \sum_{y} \dot{j}_{yb})
\]

\[
= W_a \dot{j}_a + W_b \dot{j}_b - W_a (-\dot{j}_a) - W_b (-\dot{j}_b)
\]
\[
= W_a \dot{j}_a - W_b \dot{j}_a + W_a \dot{j}_a - W_b \dot{j}_a = 2(W_a - W_b) \dot{j}_a
\]
Thomson's Principle

\( i \) is a unit Kirchhoff flow from \( a \) to \( b \)

\( j \) is any unit flow from \( a \) to \( b \)

then \( \sum i_{xy}^2 R_{xy} = \sum i_{xy}^2 R_{xy} \)

Let \( d = j - i \) then dis a zero flow \( i \) \( d_q = 0 \)

\[ \sum i_{xy}^2 R_{xy} = \sum (i_{xy} - d_{xy})^2 R_{xy} \]

\[ = \sum i_{xy}^2 R_{xy} - 2 \sum i_{xy} R_{xy} d_{xy} + \sum d_{xy}^2 R_{xy} \]

\[ \equiv \sum (V_x - V_y) d_{xy} \quad (*) \]

Set \( W = V \) & \( d = d \) then by conservation of energy

\[ (*) \equiv 4 (V_a - V_b) d_a = 0 \quad \text{thus} \]

\[ \sum i_{xy}^2 R_{xy} = \sum i_{xy}^2 R_{xy} + \sum d_{xy}^2 R_{xy} \]

\[ > \sum i_{xy}^2 R_{xy} \]
Rayleigh's Monotonicity Law

If \( \forall x, y, R_{xy} \geq R_{xy} \) then \( \overline{ER}_{ab} \geq ER_{ab} \)

Let \( \mathbf{i} = \text{unit flow from a to b in } \overline{R} \)

\[ \overline{ER}_{ab} = \mathbf{i}^2 \overline{ER}_{ab} = \frac{1}{a} \sum i_{xy}^2 R_{xy} \]

\[ \geq \frac{1}{a} \sum i_{xy}^2 R_{xy} \]

\[ \geq \frac{1}{a} \sum i_{xy}^2 R_{xy} \quad (\text{Thomson}) \]

\[ = \overline{ER}_{ab} \]