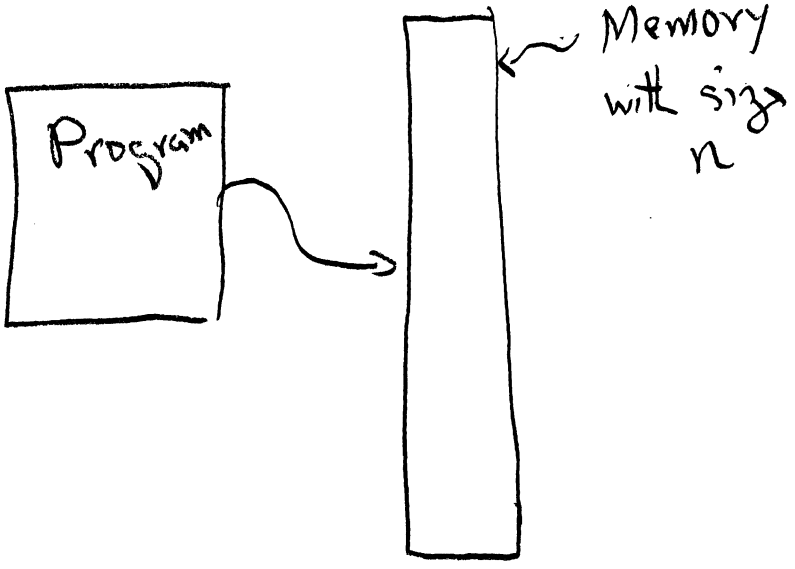
15-750
4/1/09

# Parallel Algorithms

So far we have assumed the following model: RAM model




unit time ops

$(+, \times, \div$ (log n bits)$)$

(read/write into memory)

---

A central model to describe Graph Algorithms.

Other models:

1) Agents / ants 2) pointer machines

RAM is unrealistic as $n$ goes to infinity.

1) Speed of light (large size machines)
2) quantum effects (small size machines)

---

Bottom Line: RAM

1) Many important algorithms were found using this model.
2) Most algorithms are coded in a RAM like language.
   eg C

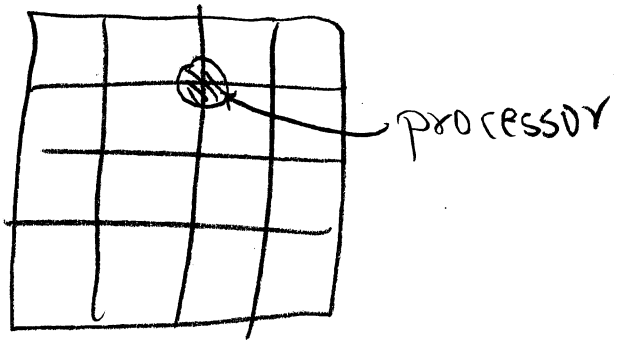
# Parallel Models

Fixed connection machines

machines ⊇ finite state machine

⊇ RAM

A) Cellular Arrays 1D, 2D, 3D




2D

1940's von Neumann

60's, 70's algorithms for CA.

Alvy Ray Smith 1974

# Highly connected models

1) Hypercube $\equiv (V, E)$ 1980's

$$V \equiv \{(a_1 \cdots a_m) \mid a_i \in \{0,1\}\} \quad m = \log n$$

$$((a_1 \cdots a_m), (a_1 \cdots \bar{a}_i \cdots a_m)) \in E$$

2) Shuffle-exchange graph 1980's

$$V = \{(a_1 \cdots a_m) \mid a_i \in \{0,1\}\}$$

$$((a_1 \cdots a_m), (\bar{a}_1 a_2 \cdots a_m)) \in E$$

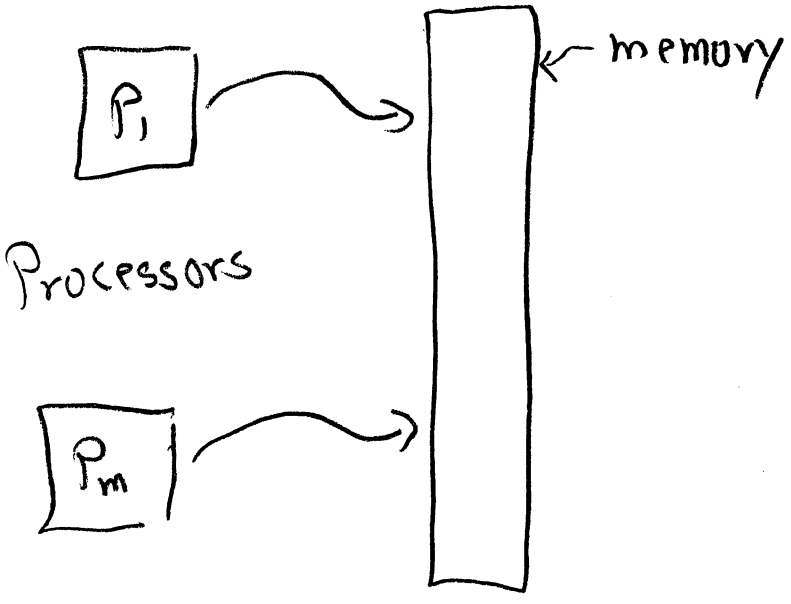
$$((a_1 \cdots a_m), (a_m a_1 \cdots a_{m-1})) \in E$$

3) Randomly connected graphs

possible models of the brain!

# Shared memory models

1) PRAM (Parallel Random Access Machine)




Unit time ops

$+, \times, \div$

read / write

| | Exclusive | Concurrent |
|---|---|---|
| Read | | |
| Write | | |

ER EW

CR CW

PRAM issues:

Penalty for CR on an ER machine?

1) Machine crashes!

2) garbage read!

---

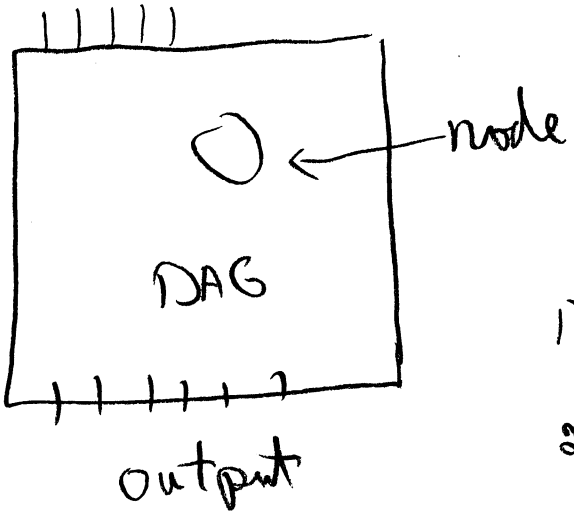
How is synchronization handled?

1) After each unit of time!

2) none!

# Circuit Model

inputs in either bits or words



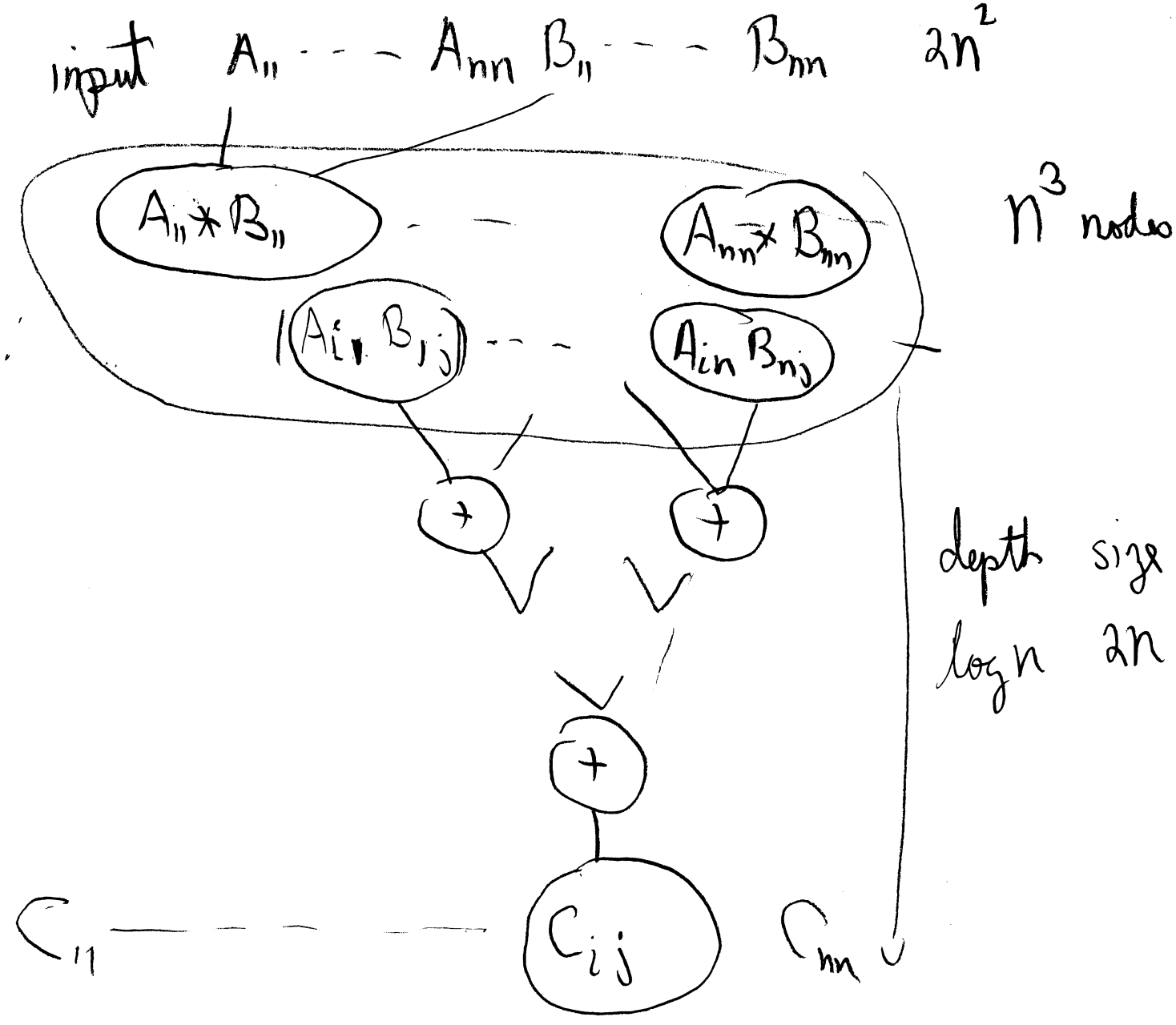

nodes 1) $\wedge, \vee, \neg$ gates
2) Arithmetic ops

1) Constant fan in.
2) Arbitray fan out.

Work $\equiv$ # nodes
Time $\equiv$ Longest path from input to output

# Naive Matrix Multiply in the Circuit Model

$$C_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj} \qquad C = A \cdot B$$



Totals: Work: $O(n^3)$

Time: $O(\log n)$

---

Naive MM on PRAM

$P \equiv$ # processor $T \equiv$ Parallel time

1) CRCW $P \equiv O(n^3)$ one processor / node

$T \equiv O(\log n)$

Note: fan out $\equiv$ reads

fan in $\equiv$ writes

2) CREW $P \equiv O(n^3/\log n)$

$T \equiv O(\log n)$

each process does the work of $\log n$ virtual process.

3) EREW $P = n^3/\log n$ $T \equiv O(\log n)$

<u>Def</u> Work $\equiv P \cdot T$

One must pay for each process for life of the run.

---

Naive MM: $O(n^3)$ work $O(\log n)$ time

Claim: If we have $P < n^3/\log n$ processors then the time $\equiv W/P$.

---

Strassen's Alg

Recall: recurrence

$$MM(n) = 7\,MM(n/2) + cn^2$$

7 recurrive calls

matrix additions

$A+B$

<u>Note</u> Matrix addition

$O(n^2)$ work

$O(1)$ time

Time: $T(n) = T(n/2) + O(1)$

parallel calls

$O(\log n)$

Work: $W(n) = 7W(n/2) + cn^2$

we must pay for each call!

$O(n^{2.81\cdots})$

# All-Prefix-Sums / Prescan

Let $\oplus$ be an associative binary op

eg $+$, $\circ$

Def: All-Prefix-Sums

input: $[a_0, \ldots, a_{n-1}]$

output: $[a_0, a_0 \oplus a_1, a_0 \oplus a_1 \oplus a_3, \ldots, a_0 \oplus \cdots \oplus a_{n-1}]$

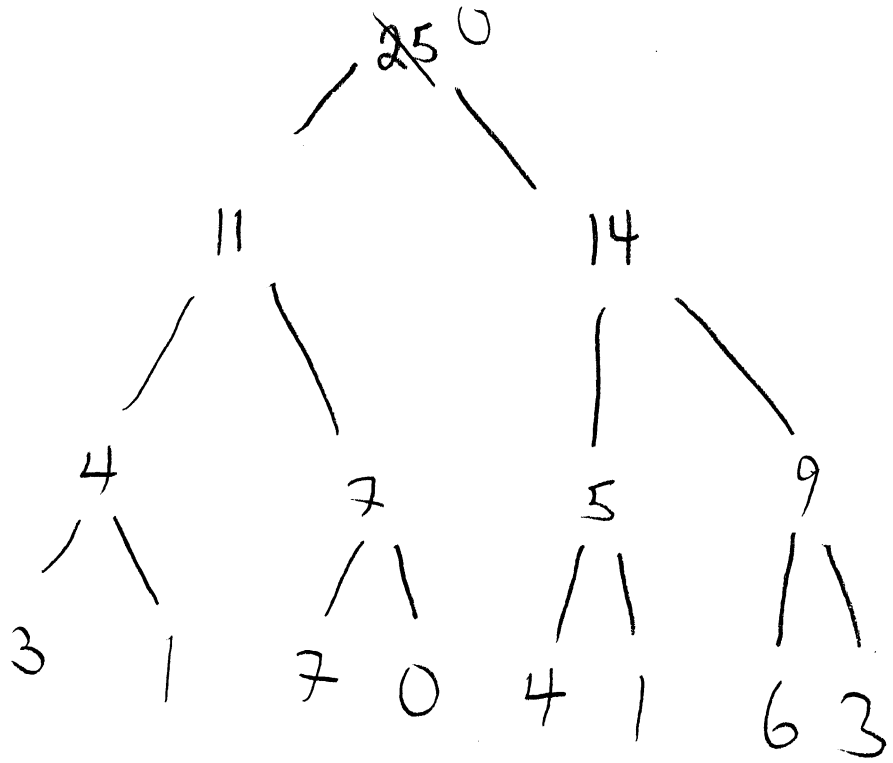
Prescan

output $[I, a_0, a_0 \oplus a_1, \ldots, a_0 \oplus \cdots \oplus a_{n-2}]$

Prescan

input: (3, 1, 7, 0, 4, 1, 6, 3) ⊕ addition

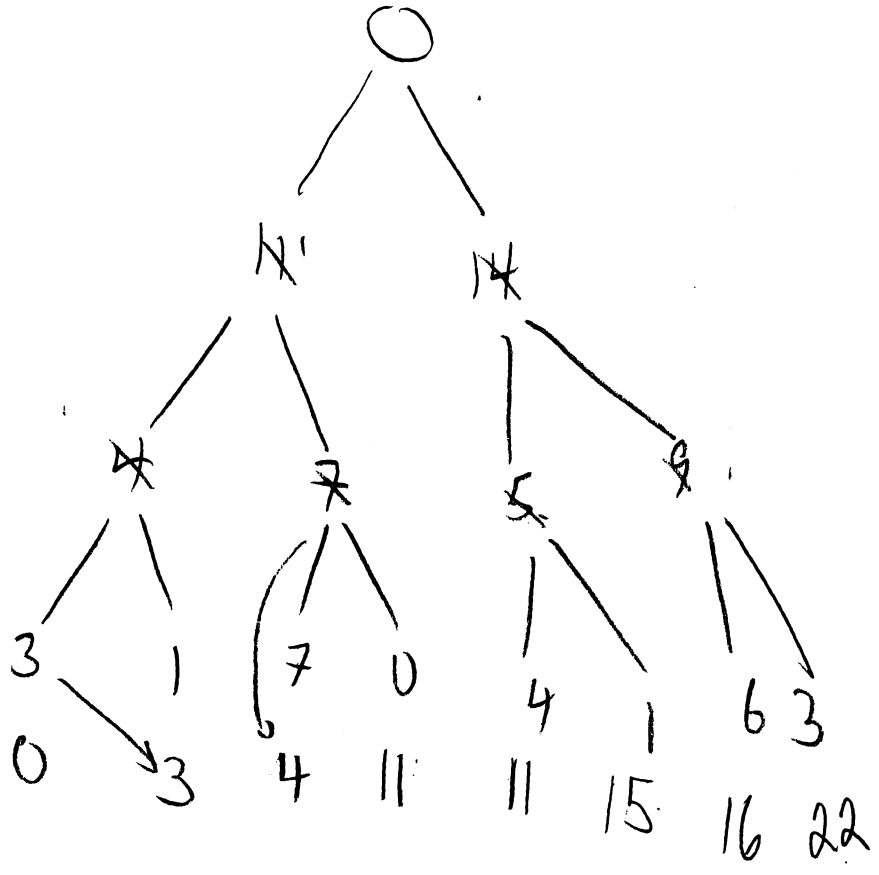
Alg 1) Compute tree of partial sums
2) set root to zero
3) DOWN!




3) Down!

a) Right-Child ⇐ Parent ⊕ Left-Child

2) Left-Child ⇐ Parent




$T(n) = O(\lg n)$

$W(n) = O(n)$ ?

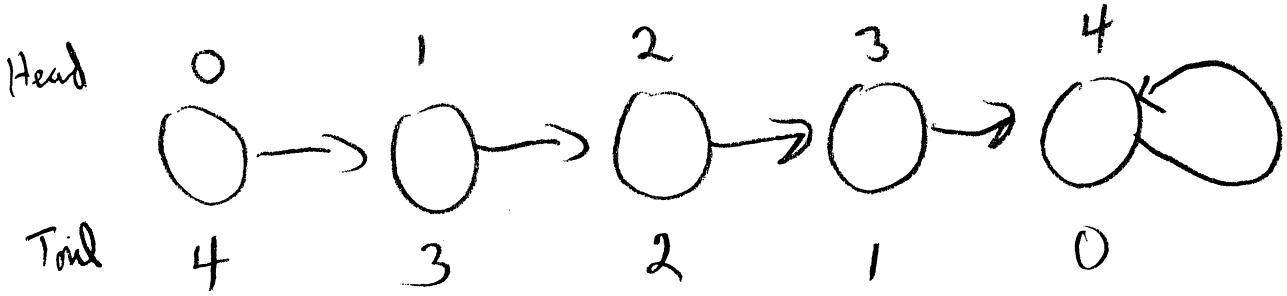
# List Ranking

input: linked-list

output: a mark on each node s.t.

mark = distance from head or

= distance to tail




Assume:

1) pointers are in consecutive memory
2) We know location of head & tail
3) pointers in arbitrary order.

# Wyllie's Alg

In|| rank(!) := 1 ; rank(tail) = 0

In|| **while** succ(head) ≠ nil **do**

  **if** succ(!) ≠ nil **do**

    rank(!) := rank(!) ⊕ rank(succ(!))

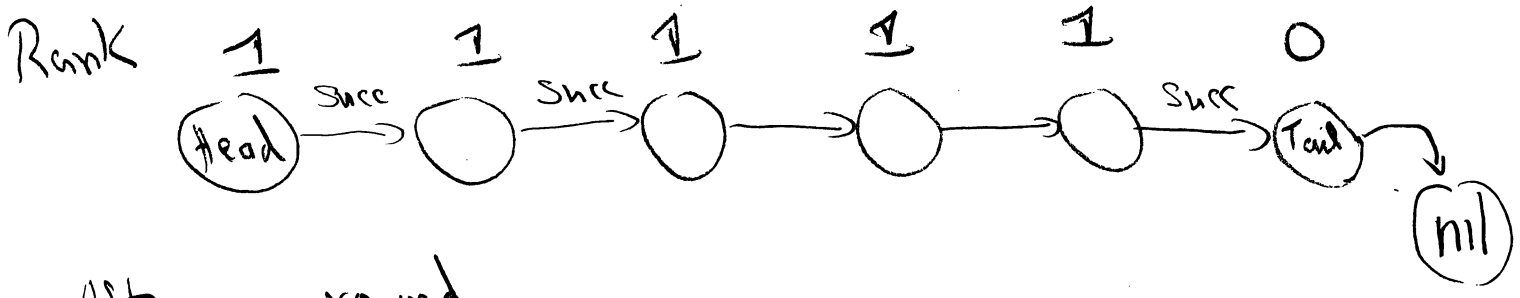
    succ(!) := succ(succ(!))

  **fi**

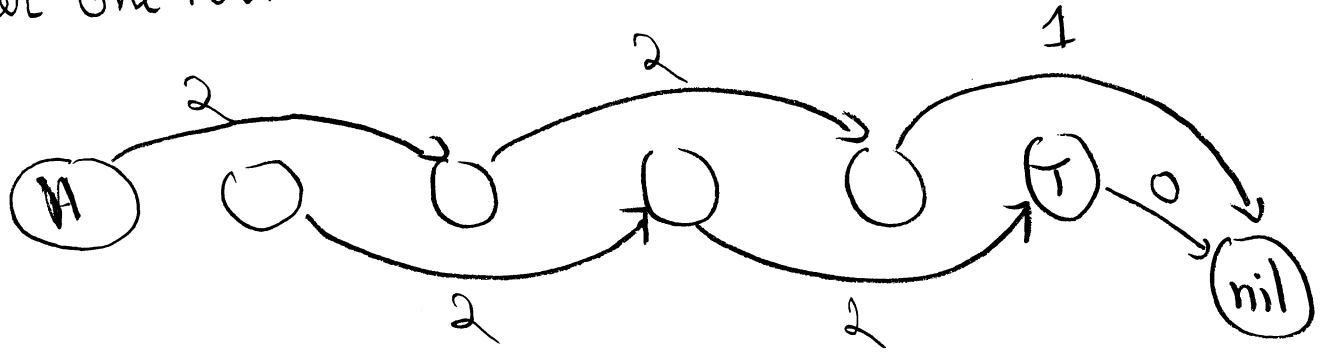
| | Goal |
|---|---|
| # processors = $n$ | |
| Time ≡ $O(\log n)$ | $O(\log n)$ |
| Work ≡ $O(n \log n)$ | $O(n)$ |
| CREW model | |

# Wyllie's Alg

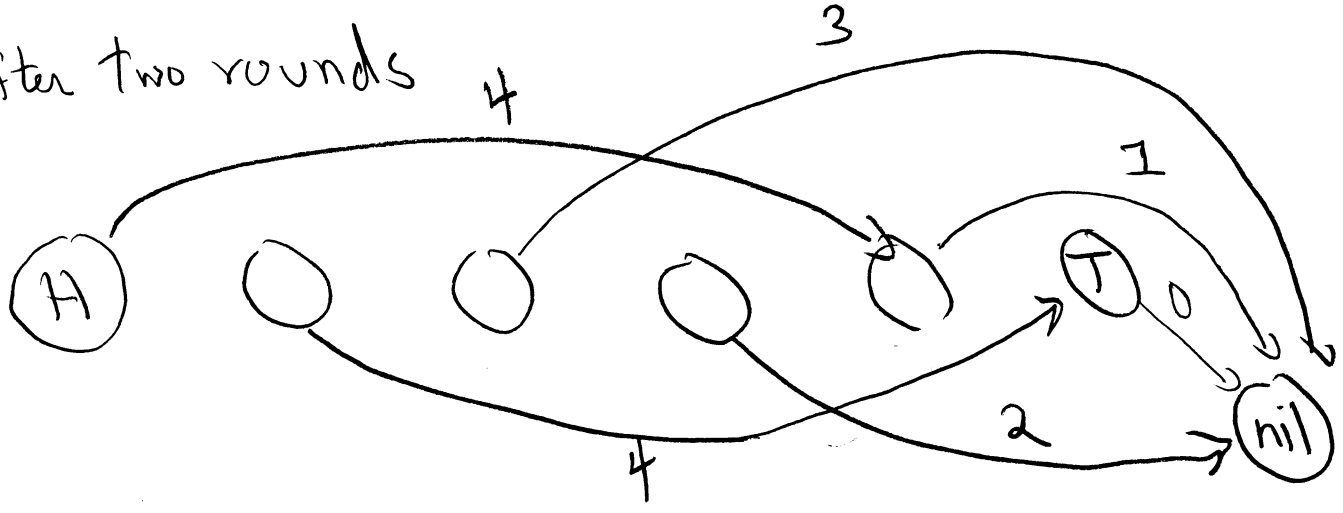
## Initial




## After one round



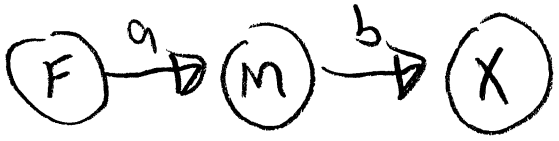

## After two rounds



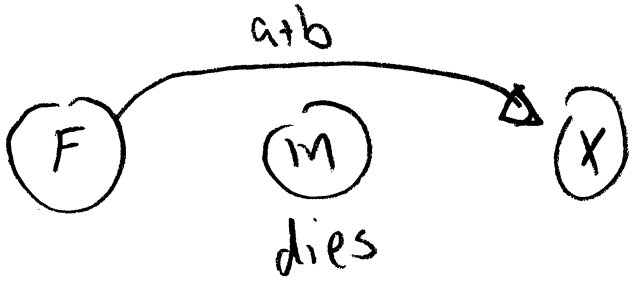
# Random - Mate

1) Contraction Phase

1) Each live node randomly picks a sex

2) If 

then




3) Stop when head points to nil.
(only head is live)

<u>Thm</u> The contraction phase stops in $c \log n$ rounds with high prob.

<u>pf</u> Let $P_i \equiv$ Event that node $i$ is still live after one round

<u>Note</u> node$_i$ not head then $\text{Prob}[P_i] = 3/4$

Let $P_i^K \equiv$ Event that node $i$ still live after $K$ rounds.

Note: $\text{Prob}[P_i^K] = (3/4)^K$ $i$ not head.

Set $K = \lfloor c \log_{4/3} n \rfloor$

$$\text{Prob}[P_i^K] = 1/(4/3)^K \leq \frac{1}{(4/3)^{c \log_{4/3} n}} = \frac{1}{n^c}$$

Let $P^k \equiv$ Event that some non-head node is still live.

Assume that $node_0$ is the head.

$$P^k = P_1^k \cup P_2^k \cup \cdots \cup P_n^k$$

$$\text{Prob}[P^k] = \text{Prob}[P_1^k \cup \cdots \cup P_n^k]$$

$$\leq \text{Prob}[P_1^k] + \cdots + \text{Prob}[P_n^k]$$

$$\leq n \cdot \frac{1}{n^c} = \frac{1}{n^{c-1}}$$

If we set $c=2$ then the contraction phase stops with prob $\leq 1/n$.

In the expansion phase we run contraction phase "backwards".

ie