Packing out memory

Input: Array of used & unused memory
Output: Array of used followed by unused memory

Input:

```
1 0 1 1 1 0 0 1 0 1 1
```

Output:

```
A A B B D C D D A B B B B
```

Solution: Use PreScan on array

\[ M_i = \begin{cases} 
1 & \text{if } i \text{th memory used} \\
0 & \text{otherwise}
\end{cases} \]
Euler Tours

Problem: Preorder numbers of a tree

Input: A tree stored using pointers

Output: Preorder numbering

Weighted List Ranking

Input: 0 1 0 1 0

Output: 0 1 1 0 1 2 2 0 2
1 = weight of down edge
0 = weight of up edge

Inorder?

Postorder?
Parallel Expression Evaluation

Example Input:

```
( +
  ( x 7 )
  ( +
    ( x )
    ( 1 4 )
  )
( 5 3 )
```

Output! Value, all subvalues

Goal: Parallel Alg

Simple Alg:

1) Assign a processor to each node.

While tree non empty do

2) if leaf "send" value to parent
   delete node [RAKE]

3) if node has 2 values then evaluate
Worst Case for simple Alg

Recall: Horner's Rule

Input: polynomial $a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$

Alg: $a_0 + x(a_1 + x(a_2 + \cdots + x(a_{n-1} + x(a_n)) \cdots))$

As a tree

Simple Alg

$O(n)$ Time
$O(n^2)$ Work
Keeping nodes with only one value busy!

Here we view the tree edges as transformers.

Init: The edge are the identity,

\[ (y+b) \xrightarrow{y} y \]

\[ a \cdot y \xrightarrow{y} a \cdot y \]

\[ a(y+b) = ay + ab \]
The general case.

\[ f(y) = ay + b \quad g(y) = cy + d \]

\[ f(g(y)) = a(ay + b) + b = acy + (ad + b) \]

Note: functions \( ay + b \) are closed under compositions.

We can also remove an independent set of 1-child nodes (degree 2 nodes).

Very similar to pivoting in Gaussian Elim.
**Def** $V_0 \ldots V_k$ is a chain if:

1) $V_{i+1}$ is only child of $V_i$, $0 \leq i < k$.
2) $V_k$ has only one child & it is not a leaf.

---

**Example**

1) Chain

---

**The Independent Set**

1) All leaves
2) Max independent set from each maximal chain
Parallel Tree Contraction

RAKE ≡ remove all leaves

COMPRESS ≡ replace each maximal chain of length \(k\) with one of length \(\frac{k}{2}\).

\(\text{CONTRACT} = \{ \text{RAKE}, \text{COMPRESS} \} \)

\[|\text{CONTRACT}(T)| \leq \frac{2}{3}|T|\]

pf

\(\text{Def } V_0 = \text{leaves of } T\)

\(V_1 \subseteq V \text{ with } 1 \text{ child}\)

\(V_2 \subseteq V \text{ with } 2 \leq \# \text{ children}\)

\(C \subseteq V_1 \text{ with child in } V_0\)
Claim \(|V_0| > |V_a|\)

**Proof** by induction on size of \(T\).

Claims: \(|V_0| > |C|\)

**Definition**
\[Ra = V_0 \cup V_a \cup C\]
\[Com = V_1 - Ra\]

\([\text{RAKE}(Ra) \subseteq V_a \cup C \Rightarrow |\text{RAKE}(Ra)| \leq \frac{2}{3} |Ra|]\)

\(\text{Note} \quad \text{Com} = \text{union of maximal chains}\)

\(|\text{COMPRESS}(Com)| \leq \frac{1}{2} |Com|\)

Cor: After \(\log_{\frac{3}{2}} n\) CONTRACTS are empty.
A Simple Randomized List-Ranking

Assume linked-list is doubly linked.

Alg Splicing-Out

1) Make $\frac{n}{\log n}$ queues of size $\log n$ (queue/proc)
2) Set sex of all nodes to $M_0$
3) Reset sex of each queue-top to random sex.
4) If top is $F$ and points to $M$ then "splice-out" top.
5) Repeat while some queue not empty.

Thm After $O(\log n)$ rounds all queues are empty with high probability.
Chernoff Bounds

Let $X_1, \ldots, X_t$ be independent 0/1 random variables.

Assume $\text{Prob}(X_i = 1) = p$.

The binomial random variable is

$$S_n^p = X_1 + \cdots + X_t$$

$$\text{Expect}(S_n^p) = \sum E(X_i) = p \cdot t$$
Thm \quad \text{Prob}(S^p_t < (1-\beta)p^t) < e^{-\beta^2 p^t/2} \quad \forall 0 \leq \beta < 1

Thm \quad \text{Prob}(S^p_t > (1+\beta)p^t) < e^{-\beta^2 p^t/2} \quad \forall 0 \leq \beta < 1
Let's fix one of the queues, say, Q.

At a given round the prob Top is spiced-out is \( \geq \frac{1}{4} \)

\[
\begin{array}{c}
\text{P} & \text{M} \\
\text{Q} & \frac{1}{2} \\
\text{Q} & \frac{1}{4}
\end{array}
\]

View prob as:

We have a coin \( \text{Prob(Head)} = \frac{1}{4} \) \( \text{Prob(Tail)} = \frac{3}{4} \)

Question: After \( t \) flips what is \( \text{Prob}[\text{#heads} < \log n] \) ?

Suppose we pick \( t \) s.t. \( \text{Expect #heads} = 4 \log n \)

ie \( t = 16 \log n \)
We apply Chernoff with 
\[ p = \frac{1}{4}, t = 16 \log n, \beta = \frac{3}{4} \]

\[
\text{Prob}\left( S_t^p < (1 - \beta) pt \right) < e^{-\beta^2 pt/2}
\]

\[
\text{Prob}\left( S_t^p < \log n \right) < e^{-\left(\frac{3}{4}\right)^2 \frac{1}{4} 16 (\log n)^2/2}
\]

\[
= e^{-9/8 \log n} = n^{-9/8}
\]

Thus \( \text{Prob} \) that some queue is not empty after \( t = 16 \log n \) rounds < \( (\log n)^{9/8} \)

< \( n^{-1/8} \)