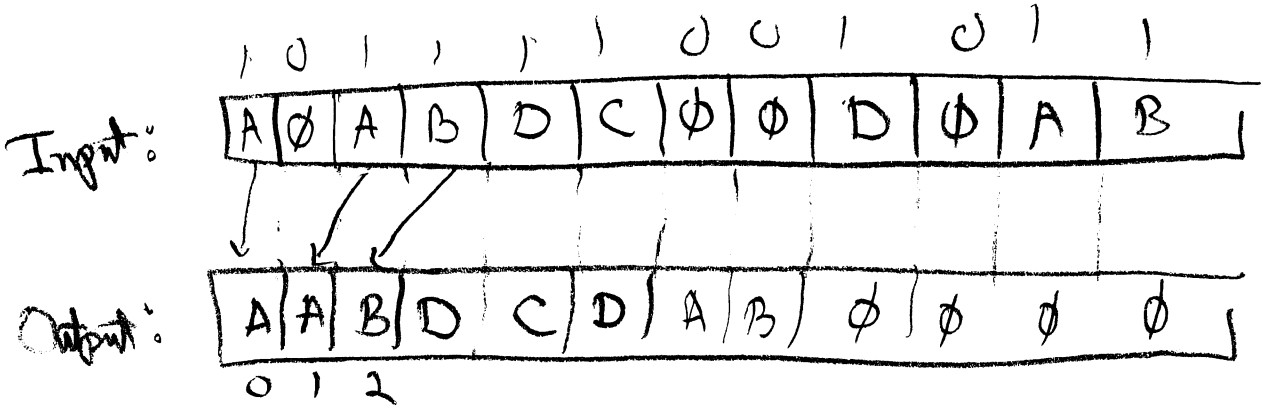
15-750
4/8/09

Parallel Algorithms II

Packing out memory

Input: Array of used & unused memory

Output: Array of used followed by unused memory



Solution: Use Prescan on array

$$M_i = \begin{cases} 1 & \text{if } i\text{th memory used} \\ 0 & \text{o.w} \end{cases}$$

Euler Tours

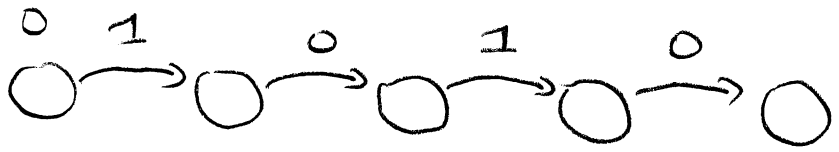
Prob: Preorder numbers of a tree

Input: A tree stored using pointers

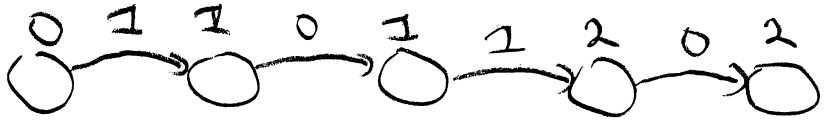
Output: Preorder numbering

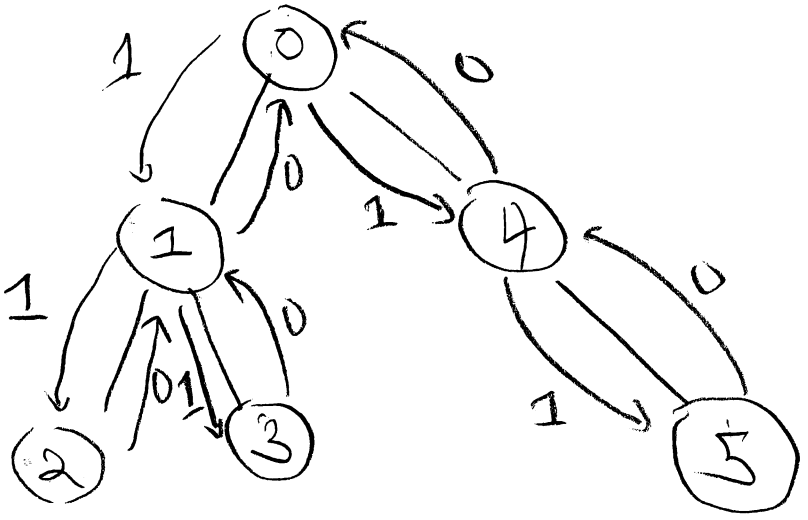
Weighted List Ranking

Input:



Output:

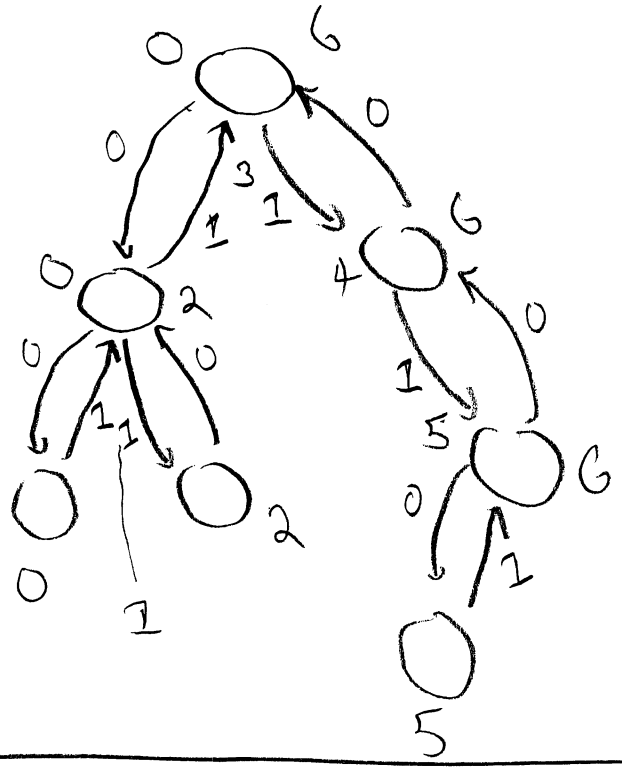




1 ≡ weight of down edge

0 ≡ weight of up edge

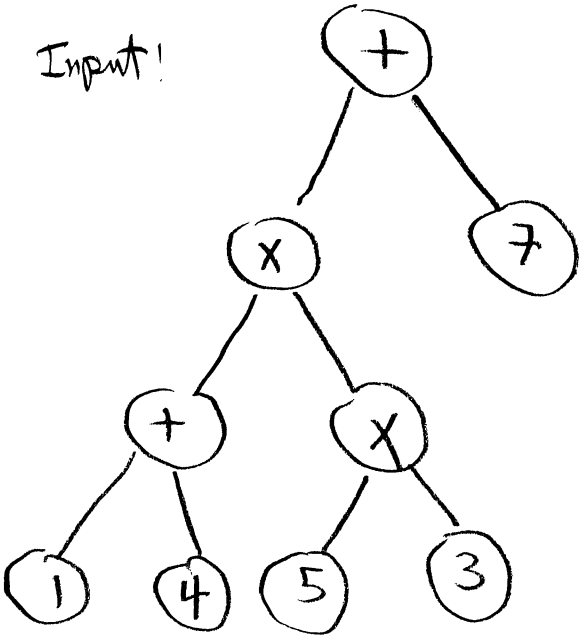
Inorder?



Postorder?

Parallel Expression Evaluation

Eg Input:



Output: Value, all subvalues

Goal: Parallel Alg

Simple Alg

1) Assign a processor to each node.

<u>While</u> tree non empty <u>do</u>

2) <u>if</u> Leaf "send" value to parent
   delete node [RAKE]

3) if node has 2 values then evalute.

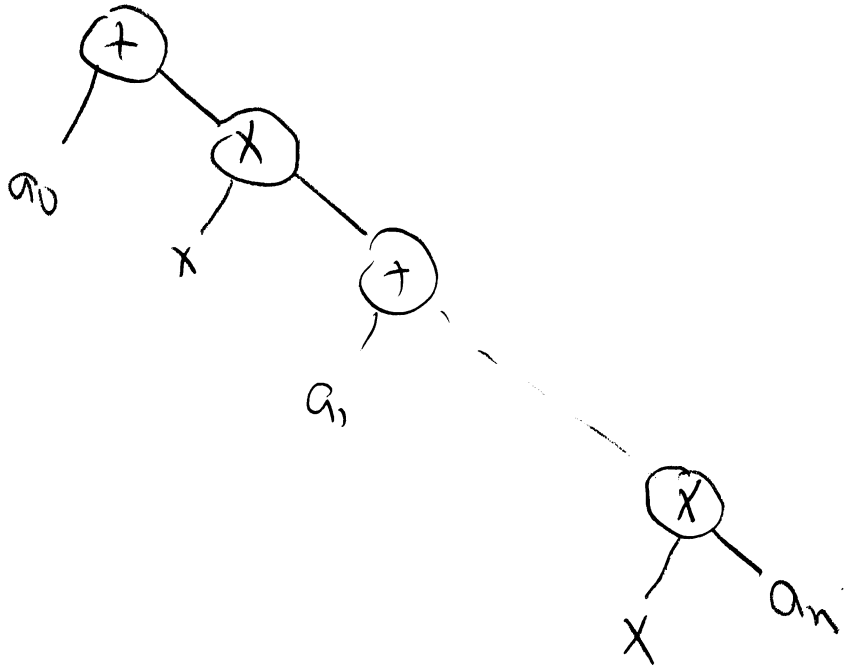
Worst Case for simple Alg

Recall: Horner's Rule

Input: polynomial $a_0 + a_1 X + a_2 X^2 + \cdots + a_n X^n$

Alg: $a_0 + X(a_1 + X(\cdots + X(a_{n-1} + X(a_n)\cdots))$

As a tree



Simple Alg

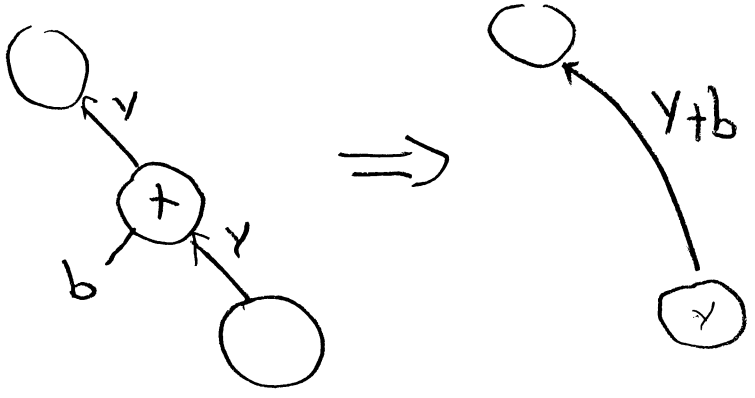
$O(n)$ Time

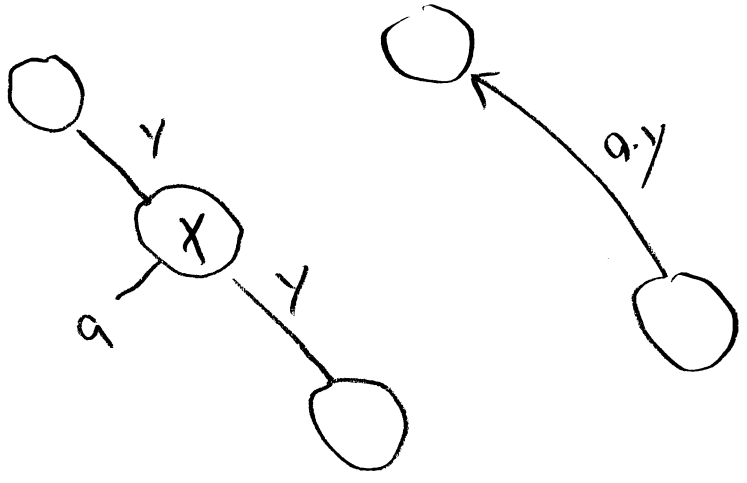
$O(n^2)$ Work

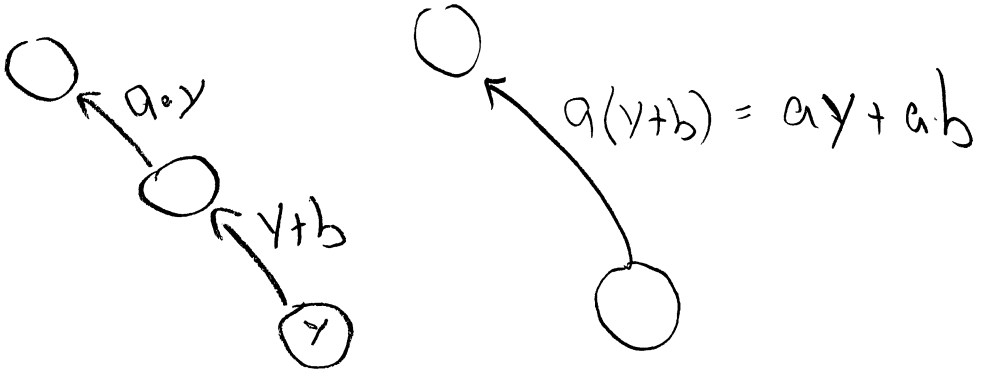
Keeping nodes with only one value busy!

Here we view the tree edges as transformers.

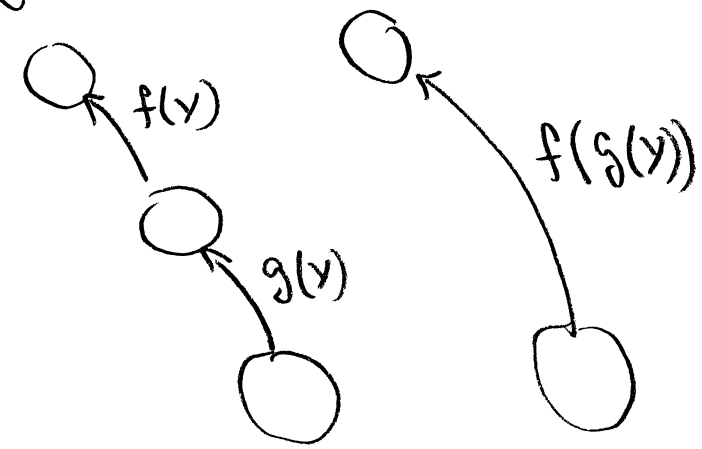
Init: The edge are the identity.







The general case.



$f(y) = ay + b \qquad g(y) = cy + d$

$f(g(y)) = a(cy+d) + b = \overline{a \cdot c}\, y + (ad + b)$

Note: fcns $ay+b$ are closed under compositions

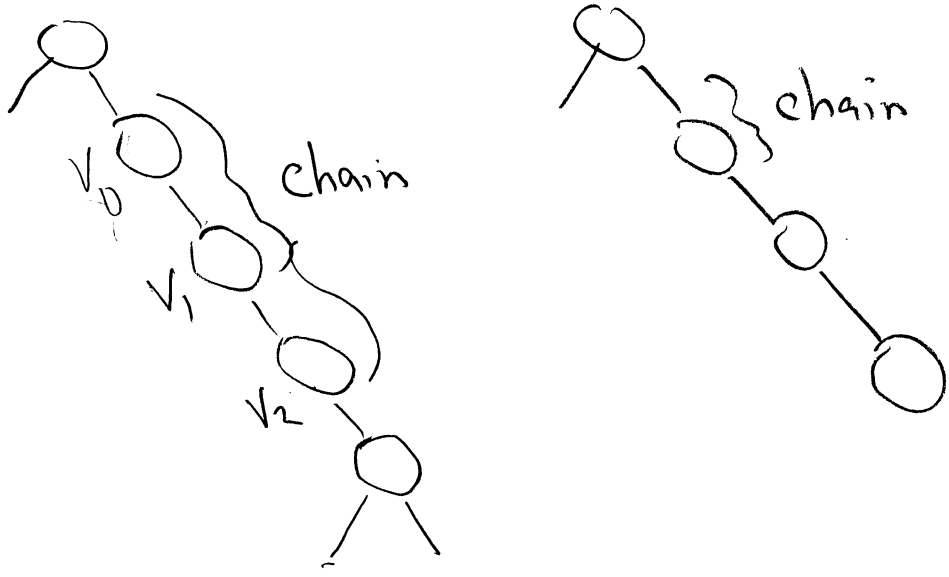
We can also remove an independent set of 1-child nodes (degree 2 nodes)

Very similiar to pivoting in Gaussian Elim!

<u>Def</u> $V_0 \cdots V_K$ is a <u>chain</u> if:

1) V_{i+1} is only child of V_i $0 \le i < K$.
2) V_K has only one child & it is not a leaf

Eg:



The <u>Independent Set</u>

1) All leaves
2) Max independent set from each maximal chain

Parallel Tree Contraction

RAKE $\equiv$ remove all leaves

COMPRESS $\equiv$ replace each maximal chain of length k with one of length $\lfloor k/2 \rfloor$.

CONTRACT $\equiv \{$RAKE, COMPRESS$\}$

<u>Thm</u> $|\text{CONTRACT}(T)| \leq 2/3\,|T|$

<u>pf</u> <u>Def</u> $V_0 =$ leaves of T

$V_1 \subseteq V$ with 1 child

$V_2 \subseteq V$ with $2 \leq$ # children

$C \subseteq V_1$ with child in V_0

<u>Claim</u> $|V_0| > |V_2|$

<u>Pf</u> induct on size of T.

Claim: $|V_0| \geq |C|$

<u>Def</u> $R_a = V_0 \cup V_2 \cup C$

$Com = V_1 - R_a$

$RAKE(R_a) \subseteq V_2 \cup C \Rightarrow |RAKE(R_a)| \leq 2/3 |R_a|$

<u>Note</u> $Com \equiv$ union of maximal chains

$|COMPRESS(Com)| \leq \frac{1}{2} |Com|$

<u>Cor</u> After $\log_{3/2} n$ CONTRACTS T_{∞} empty

A Simple Randomized List-Ranking

Assume Linked-list is doubly linked.

Alg Splicing-out

1) Make $n/\log n$ queues of size $\log n$ (1 queue/proc)
2) Set sex of all nodes to M.
3) Reset sex of each queue-top to random sex.
4) In ll <u>If</u> top is F and points to M <u>then</u> "splice-out" top.
5) Repeat while some queue not empty.
   3) & 4)

<u>Thm</u> After $O(\log n)$ rounds all queues are empty with high probability.

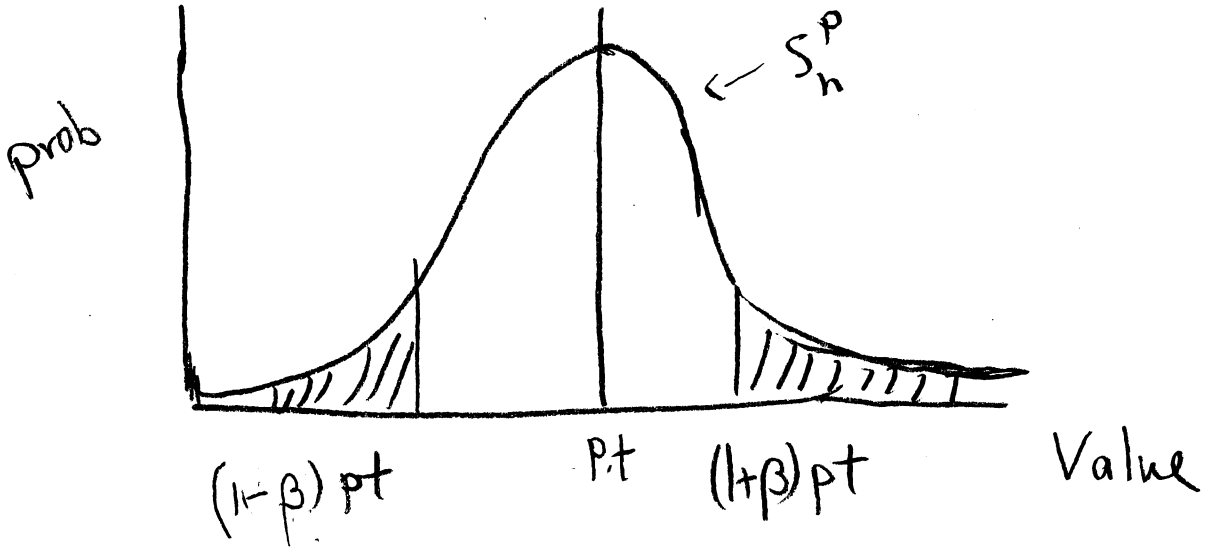
Chernoff Bounds

Let $X_1 \cdots X_t$ be independent 0/1 random variables

Assume $\text{Prob}(X_i = 1) = p$

The binomial random variable is

$$S_n^p = X_1 + \cdots + X_t$$



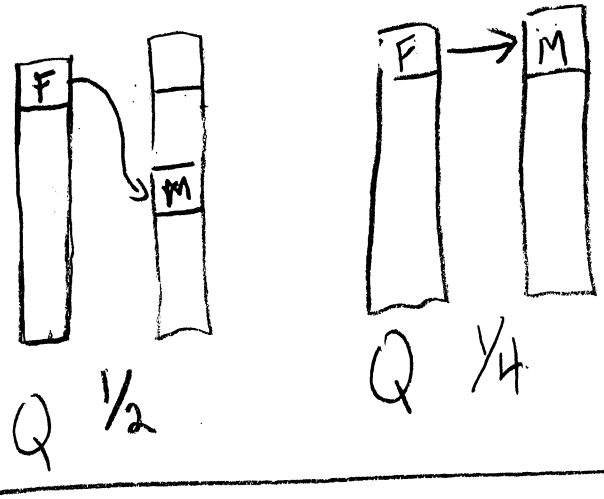
$$\text{Expect}(S_t^p) = \sum E(X_i) = p.t$$

<u>Thm</u> $\text{Prob}(S_t^p < (1-\beta)pt) < e^{-\beta^2 pt/2} \quad \forall\, 0 \le \beta < 1$

<u>Thm</u> $\text{Prob}(S_t^p > (1+\beta)pt) < e^{-\beta^2 pt/3} \quad \forall\, 0 \le \beta < 1$

Lets fix one of the queues, say Q.

At a given round the prob Top is spliced-out is $\geq 1/4$



View prob as:

We have a coin Prob(Head) = 1/4 Prob(Tail) = 3/4

Question: After t flips what is

Prob[#heads < log n] ?

Suppose we pick t s.t. Expect #heads = 4 log n

ie $t = 16 \log n$

We apply Chernoff with

$p = 1/4$, $t = 16 \log n$ $\beta = 3/4$

$$\text{Prob}\left(S_t^p < (1-\beta)pt\right) < e^{-\beta^2 pt/2}$$

$$\text{Prob}\left(S_t^p < \log n\right) < e^{-(3/4)^2 (\frac{1}{4}) 16 (\log n)/2}$$

$$= e^{-9/8 \log n} = n^{-9/8}$$

Thus Prob that some queue is not empty after $t = 16 \log n$ rounds $< (n/\log n)\, n^{-9/8}$

$$< n^{-1/8}$$