Claim: An assignment $A$ making $(\bar{a} \lor b \lor c \lor d)$ true.

Ill $T$ plus assignment $A$ making $(\bar{a} \lor b \lor c \lor d)$ true, Clause $(\bar{a} \lor b \lor c \lor d)$ and new variable $x$.

Idea: Claim $(\bar{a} \lor b \lor c \lor d)$ true.

Show CNF $\leq_\text{p} 3\text{CNF}$

Then 3CNF is NP-Complete

$3CNF = \{ \varphi \in CNF \mid \text{at most 3 literals per clause} \}$

NP-Completeness (cont)

15-25D 4/20/09 cont
\( f : \text{CNF} \rightarrow 3 \text{CNF} \)

Replace each clause \((a_1 \lor \ldots \lor a_t)\) with
\[(a_1 \lor a_2 \lor x_1) (\bar{x}_1 \lor a_3 \lor x_2) (\bar{x}_2 \lor a_4 \lor x_3) \ldots (\bar{x}_{t-3} \lor a_t)\]
where \(x_1, \ldots, x_{t-3}\) are new variables

We need to check
1) \( f \) is Poly-time (easy)
2) \( \Phi \) is Sat iff \( f(\Phi) \) is Sat

2) \((\Rightarrow)\quad T(\Phi) = \text{True} \quad (a_1 \lor \ldots \lor a_t) \text{ clause of } \Phi \)

Some \(i \quad T(a_i) = \text{True} \)

Set \( T(x_j) = \text{True} \) for \( j \leq t-2 \)
\( T(x_j) = \text{False} \) for \( j > t-2 \)

\[(a_1 \lor a_2 \lor x_1) (\bar{x}_1 \lor a_3 \lor x_2) \ldots (\bar{x}_{t-3} \lor a_t) \lor x_{t-1}) \ldots = \text{CNF}^*\]
2) $\Leftarrow$ Suppose $T(C^*) = \text{True}$

$C^*$ has $t-2$ True clauses

Claim Some literal $a_i$ s.t. $T(a_i) = \text{True}$

Matching Prob

\[
\begin{array}{c|c|c}
\text{New} & x_i & 3-\text{Clauses} \\
\text{Variables} & & (\text{All True}) \\
\hline
\text{t-3} & \text{t-2}
\end{array}
\]

\[\therefore \text{Some 3-Clause all new literals are False}
\]

Thus Original literal must be true.

\[(a_1 \lor \neg a_t) \equiv C\]
Def. A **clique** is a completely connected subgraph.

Clique = \{ (G,k) | G is a graph with a k-clique \}

Thm. Clique is NP-Complete

Note. Clique ∈ NP

Suffice to show: CNF ≤p Clique

⇒ f: cnf formula → (Graph, integer)

Suppose \( \Phi \) has clauses \( C_1, \ldots, C_k \)

literals \( \alpha_1, \ldots, \alpha_m \)

Def. \( f(\Phi) = (G=(V,E), k) \)
\[ V = \{(a_i, C_j) \mid a_i \text{ is a literal in clause } C_j \} \]

\[ \bar{E} = \{(a, C), (a', C') \mid C \neq C' \text{ and } a \neq \overline{a'} \} \]

Claim: \( \phi \in \text{CNF} \iff f(\phi) \in \text{Clique} \)

\[ \Rightarrow \text{ Suppose } T(\phi) = \text{True} \]

(At least one true literal per clause)

Say \( a_i \) is true in \( C_i \)

Then \( \{(a_i, C_i), \ldots, (a_k, C_k)\} \) is a clique in \( G \).

\[ \Leftarrow \text{ Suppose } H \text{ is a } k \text{-clique in } G \]

\[ m \{ (a, C), \ldots, (a_k, C_k) \} \]

1) \( C_i \)'s are distinct

2) No \( a_i = \overline{a_j} \)
Set \( T(x) = \begin{cases} 
True & \text{if } x = 0, i \text{ some } i \\
False & \text{otherwise}
\end{cases} \)

**Def** \( I \subseteq V \) is independent if \( \forall x, y \in I \ (x, y) \notin E \).

**The Independent Set Prob**

\[\text{IND} = \{ (G, k) \mid \text{G has an Ind set of size } k \}\]

**Thm** \(\text{IND is NP-Complete}\)

\(\text{IND} \in \text{NP}\)

\(\overline{G} = (V, \overline{E})\)

\[\overline{E} = \{ (v, w) \in \overline{E} \mid (v, w) \notin E \}\]

\[f: (G, k) = (\overline{G}, k)\]
Vertex Cover (VC)

Input: $G = (V, E)$, integer $k$

Question: $\exists C \subseteq V \mid |C| \leq k$

$C$ contains at least one end-point of each edge

Thm: VC is NP-Complete

To show $\text{IND} \leq_p \text{VC}$

$f[(G, k)] = (G, n-k)$

Note: $S \subseteq V$ is in ind iff $V - S$ is a vertex cover.
Two Graph Problem

\[ G = (V,E) \]  undirected (directed)

Eulerian cycle visits each edge exactly once.

Hamiltonian" "vertex" "

Def. \( G \) is Eulerian if \( G \) has an Eulerian cycle

\( G \) is Hamiltonian

Thm. \( G \) is Eulerian iff 1) \( G \) is connected

2) Every vertex has even degree
DHC = Directed Hamiltonian Circuit Prob.

Thm: DHC is NP-Complete

Vertex Cover \leq_p DHC

Gadgets

Consider

Only two "wirings"

Straight through

Laced
The construction of $G' = f(G, k)$  

$G$ is undir

$G'$ is directed

The pieces of $G'$

For each $E = (v, w)$ of $G$

Make

\[
\begin{array}{c}
\Downarrow \\
\hline
\Rightarrow \\
\hline
\Uparrow \\
\end{array}
\]

Plus vertices $a_1, \ldots, a_k$ (one for each "cover vertex")

Connecting the pieces:

For each $V \in G$ say
Claim: $G$ has a $k$-cover iff $G'$ is Hamiltonian

$(\Rightarrow)$ Suppose $\{u_1, \ldots, u_k\}$ is a vertex cover. Use edges $u_i \rightarrow u_i$.

If $v$ also in cover then go straight thru else lace

Note: $\forall \{u_1, \ldots, u_k\} \subseteq V$ we get a simple cycle missing uncovered edges.

If $\{u_1, \ldots, u_k\}$ is a cover then it is Ham.

$(\Leftarrow)$ Similar