15-750 — Graduate Algorithms — Spring 2006

Miller and Derryberry

Assignment 3 Due date: Monday, March 20, 2006.

Some Reminders:

- Read the Policies section on the course website before you start working on this assignment.
- You may work in **groups of size up to 3** for this problem set if you wish. However, **you should write up your solutions separately**. That is, collaboration should be limited to talking about the problems, so that your writeup is written entirely by you and not copied from your partner. In addition, state whether you worked alone and **list all collaborators**.
- Please refrain from consulting external materials when solving these problems. This does not apply to looking up standard inequalities, definitions, and such things (e.g., $(1 + 1/x)^x \leq e$ or Stirling's Formula).
- Please **submit both an electronic version, as well as a hard copy of your solutions** at the beginning of class on the due date.
- In all problems, it is implicit that you should show that your answer is correct, even when this is not explicitly stated.
- If you have questions, contact the course staff.

1 Finding Bottlenecks

(15 points) Let $G = (V, E, w)$ be an undirected and weighted graph. Recall that the **bottleneck** of a path is the weight of the minimum weight edge on the path.
Give an $O(n + m)$ time algorithm for finding the maximum bottleneck path in G from a given vertex s to a given vertex t.

2 Outer Planar Graphs

We say that a graph is outer-planar if it can be embedded in the plane so that all the vertices appear on the outer face.

(a) (5 points) Show that every outer-planar graph G has a 2-vertex separator. (Recall that, unless specified otherwise, a separator must partition the graph so that vertices can be placed into 2 sets such that there are no edges between the two 2 sets, and neither set is larger than $2n/3$.) Show how to find this separator of G in $O(n)$ time given the outer-planar embedding.

A significant amount of credit will be given if you can solve the special case in which G is 2-connected (i.e., each pair of vertices has 2 vertex-disjoint paths connecting them), although with a bit more work, you can generalize this to the case in which the graph has articulation points and multiple connected components.

(b) (5 points) Show how to find three vertices whose removal separates the outer-planar graph into connected components of size at most $n/2$.

(c) (5 points) Give a worst-case upper-bound $u(n)$ and lower-bound $\ell(n)$ for the size of an edge separator for an outer-planar graph with n vertices. In other words:

- Find the smallest function $u(n)$ such that, given an outer-planar graph G_n with n vertices, we can always separate G_n into two components of size at most $2n/3$ by cutting at most $u(n)$ edges.
- Find the largest function $\ell(n)$ such that for each $n \in \{1, 2, \ldots\}$, a graph G_n with n vertices exists such that no set of edges smaller than $\ell(n)$ separates G_n into components of size at most $2n/3$.

3 Planarity of a Hamiltonian Graph

A graph is said to be **Hamiltonian** if it contains a simple cycle containing all the vertices. An undirected graph is **cubic** if the degree of every vertex is three. Suppose graph G is a cubic Hamiltonian graph where the cycle is just the vertices in the order $(v_1, \ldots, v_n)$.

(a) (10 points) Show how to test planarity of G in $O(n^2)$ time.

(b) (5 points) If G is not planar show how to find a minimal subgraph that is not planar in $O(n^2)$ time.

(c) (Extra Credit) Find an asymptotically better algorithm to test the planarity of G.