15-750 — Graduate Algorithms — Spring 2006

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Assignment 0 Due date: Friday, January 27, 2006.

Some Reminders:

- Read the Policies section on the course web site before you start working on this assignment. Collaboration is not permitted for this assignment. Also, you should refrain from using outside sources when solving these problems. For each problem, state whether you have seen it before. If you have questions, contact the course staff.

1 Asymptotic Notation

[10 points] Order the following functions by increasing order of asymptotic growth: $(\log n)^{\log n}$, $(\log n)^{\sqrt{n}}$, $(n^{100})^{(n^{100})}$, $n!$, $(\frac{n}{2})^n$, $n^{\frac{n}{2}}$, $(\log n)^{((\log n)^{\log n})}$. Provide a brief argument justifying each successive step in the ordering.

2 Probability

[5 points] Suppose we are going to roll two 6-sided fair dice. Let E_1 be the event that the first die roll is even, and E_2 be the event that the sum of the two dice rolls is 9. Are E_1 and E_2 independent? Provide an argument for why your answer is correct.

3 Geometry

[5 points] Suppose we have three noncollinear points, p_1, p_2, and p_3, in 3-dimensional space. The *affine combination* of these points is the set of points $\alpha_1 p_1 + \alpha_2 p_2 + \alpha_3 p_3$ for which $\sum_{1 \leq i \leq 3} \alpha_i = 1$. What shape is the affine combination of these points? The *convex combination* of these points is the set of points $\alpha_1 p_1 + \alpha_2 p_2 + \alpha_3 p_3$ for which $\sum_{1 \leq i \leq 3} \alpha_i = 1$ and $\alpha_i \geq 0$ for $1 \leq i \leq 3$. What shape is convex combination of these points? If there were more than 3 points, what is the convex combination of the points?[1]

4 Upper and Lower Bounds

Consider the problem of finding an element or elements of a certain rank or ranks from an unsorted set of n distinct, comparable elements. We define the cost of an algorithm for such a problem to be the number of comparisons it makes. You are to provide the best possible upper-bound or lower-bound for the number of comparisons that the optimal algorithm makes. Your grade will be based on the *exact* value of your bound (e.g., $7n/4$), *not* the value of your bound in O-notation (e.g.,

[1] Technically, we are referring to the *set* of affine combinations and convex combinations here.

$O(n)$). Grading guidelines will be given below each problem, but bounds listed in the guidelines are not necessarily achievable. If you are unsure of your solution for the best bound, feel free to submit up to one solution for each stated bound, and we will give you points according to the best bound you correctly prove.

(a) Prove an upper-bound for the problem of finding both the largest element and the smallest element simultaneously.

- 5 points total for $2n + O(1)$
- 10 points total for $3n/2 + O(1)$.
- 15 points total for $n + \log_2 n + O(1)$.

(b) Prove an upper-bound for the problem of finding the second largest element.

- 5 points total for $2n + O(1)$
- 10 points total for $3n/2 + O(1)$.
- 15 points total for $n + \log_2 n + O(1)$.

(c) Prove a lower-bound for the problem of finding the second largest element.[2]

- 5 points total for $n - O(1)$
- 10 points total for $n + \log_2 n - O(1)$.
- 15 points total for $3n/2 - O(1)$.

5 Picket Fence Painting

[10 points] Suppose we want to paint a picket fence with n consecutive pickets. Each picket has a prescribed color it is to be painted. In a single step or stroke we can paint any number of consecutive pickets in a single color. If a picket is painted more than once, then the last color to be painted on it is what appears on the picket. Give a dynamic programming algorithm to determine the minimum number of strokes needed to paint the fence. Make sure to include a proof of correctness with your algorithm.[3]

Hint: Show how to simplify the type of strokes needed in any coloring. We can see an $O(n^3)$ algorithm.

[2] Edited from $n - \Omega(1)$ to $n - O(1)$, etc.

[3] This problem is based on a copyrighted programming contest problem from TopCoder, Inc.