Foundations of Dataflow Analysis

15-745 Optimizing Compilers
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Ingredients of a dataflow analysis

* direction: forward or backward
* flow/transfer function
* combining ("meet") operator
* dataflow values

A thought experiment

* As a thought experiment, consider programs with just one flow value
  * e.g., for reaching definitions, a program containing just one definition
  * e.g., for liveness analysis, a program with just one variable
  * e.g., for available expressions, a program with just one expression

Reaching definitions, again

combining operation (aka dataflow "meet" operator)

$$in(n) = \bigcup_{p \in pred(n)} out(p)$$

$$out(n) = gen(n) \cup (in(n) - kill(n))$$

flow function

$$out_n(i) = gen_n \vee (i - kill_n)$$

Dataflow values

* For such simple programs, in(n) and out(n) would be elements of the set {0,1}, assuming a bit-vector representation
  * for RD: 1 = the def reaches
  * for LV: 1 = the var is live
  * for AE: 1 = the expr is available

Combining operation

* When two or more control paths meet at a single node, the dataflow values from each path must be combined, according to the in() equation (for forward analyses) or out() equation (for backward analyses)
  * RD: union / bit-wise $\vee$
  * LV: union / bit-wise $\vee$
  * AE: intersection / bit-wise $\wedge$

The "meet" operator

Recall: lattices

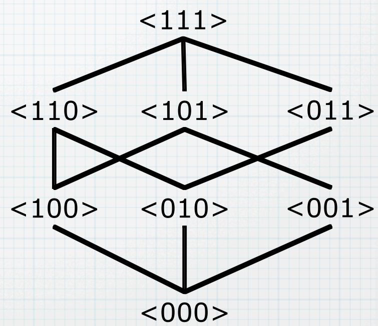
* A **lattice** L is a (possibly infinite) set of values, along with $\sqcup$ and $\sqcap$ operations
  * $\forall x,y \in L$, $\exists$ unique w and z such that
    * $x \sqcup y = w$ and $x \sqcap y = z$
  * $\forall x,y \in L$, $x \sqcup y = y \sqcup x$ and $x \sqcap y = y \sqcap x$
  * $\forall x,y,z \in L$, $(x \sqcup y) \sqcup z = x \sqcup (y \sqcup z)$ and $(x \sqcap y) \sqcap z = x \sqcap (y \sqcap z)$
  * $\exists \perp, \top \in L$, such that $\forall x \in L$,
    * $x \sqcup \top = \top$ and $x \sqcap \perp = \perp$

Two-point lattice

* So, in fact, the values {0,1}, along with the bit-wise operations $\vee$ and $\wedge$ form a two-point lattice
  * $\vee$ is the $\sqcup$ ("join") operator
  * $\wedge$ is the $\sqcap$ ("meet") operator

Dataflow values are lattice elements

- For our simple single-dataflow-value programs, the dataflow values are elements of the two-point lattice
- And this is easily generalized to bit-vectors of arbitrary (but fixed) length



The lattice BV^3

Monotonic functions

- The join and meet operations induce a **partial order** on the lattice elements
  - $x \sqsubseteq y$ if and only if $x \sqcap y = x$
  - reflexive, anti-symmetric, transitive
- For a lattice L, a function f: L→L is **monotonic** if for all x,y∈L,
  - $x \sqsubseteq y \Rightarrow f(x) \sqsubseteq f(y)$
  - i.e., larger args give larger results

RD is monotonic

$$out_n(i) = gen_n \vee (i - kill_n)$$

- Claim: The flow function for reaching definitions is monotonic
- Proof (for single-value programs) by contradiction:
  - Suppose $out_n(1) = 0$, for some n
  - Then $gen_n=0$ and $kill_n=1$
  - But $kill_n=1$ only if n is a definition, which would mean that $gen_n=1$

In dataflow analysis, we require that all flow functions be monotone

Fixed points

* Given lattice L and function $f:L\rightarrow L$, a **fixed point** of f is an element $z \in L$ such that $f(z)=z$
* Furthermore, assuming there are no infinite ascending chains in L, iteration of f starting from $\perp$ is guaranteed to find a fixed point
  * an **effective chain** wrt f is a sequence of lattice elements, $[f(i), f(f(i)), f(f(f(i))), \ldots]$ for some $i \in L$

In dataflow analysis, we require that all flow functions be monotone and have only finite-length effective chains

Dataflow analysis framework

* A control-flow graph with nodes $\{0,1,\ldots,m\}$,
* for each node, a corresponding monotonic flow function over lattice L, $\{f_0,f_1,\ldots,f_m\}$,
* a combining (aka "meet") operator, $\otimes$, and
* an initial value, i_0, giving the lattice value of the entry (or exit) block(s)

Iterative dataflow analysis

* Then **forward iterative dataflow analysis** is the least solution in L to these equations:

$$\boxed{\begin{aligned} &in(entry) = i_0 \\ &in(n) = \bigotimes_{p \in pred(n)} out(p), \quad \text{for non-entry nodes} \\ &out(n) = f_n(in(n)) \end{aligned}}$$

For RD, LV, and AE, the f_n are monotonic and have finite effective chains, so the **iterative method will work**

Similarly for backward iterative dataflow analysis

Questions

* Is it sound (i.e., are the results conservative)?
* How good are the results?
* How fast does it run?

How good is it?

* Consider a path of execution that starts from the **entry** node and ends at node **n**:
  * **entry** $\rightarrow$ **b_1** $\rightarrow$ **b_2** $\rightarrow$ ... $\rightarrow$ **n**
* The dataflow information for this path is given by composing the flow functions:
  * $f_n(\ldots(f_{b_2}(f_{b_1}(f_{entry}(i_0))))\ldots)$

MOP

* Then the "best" possible solution to a dataflow problem for node **n** is given by computing the dataflow information for all possible paths from **entry** to **n**, and then combining them with $\otimes$
  * in general there will be an infinite number of possible paths to **n**
* This is called the **meet-over-all-paths solution**

IDA is conservative

* Thm [Kildall73]: The MOP solution for block n is $\sqsubseteq$ the solution given to in(n) by iterative dataflow analysis
  * a related theorem by [Cousot&Cousot76]
* In other words, the iterative dataflow analysis solution is a conservative approximation to the "best" solution

Distributive lattices

* BV^n is a **distributive lattice**
  * L is distributive if $\forall x,y,z \in L$,
    * $(x \sqcup y) \sqcap z = (x \sqcap z) \sqcup (y \sqcap z)$, and
    * $(x \sqcap y) \sqcup z = (x \sqcup z) \sqcap (y \sqcup z)$

BV^n analyses are precise

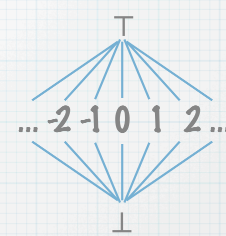
* Thm [Kildall73]: For iterative dataflow analysis over a distributive lattice, in(n) is the MOP solution for *n*
* Thus, reaching definitions, liveness, and available expressions are all solved precisely by iterative dataflow analysis

Not all DAs are distributive

* There are important dataflow analyses that use lattices that are not distributive
* Consider, for example, constant-propagation analysis...

Constant-propagation analysis

* For a single-variable program:
  * direction: forward
  * dataflow value: element of the CP-lattice
  * meet operator: CP-lattice $\sqcup$
  * initial entry value: $\perp$



CP-lattice

$\perp$ means "uninitialized variable"

$\top$ means "not a constant"

Constant-propagation analysis

- For multi-variables we have the lattice CP*:
  - elements: functions f: Var → CP-lattice
  - $\bot_{CP^*}$: $\bot_{CP^*}(v)=\bot$, all v
  - $\top_{CP^*}$: $\top_{CP^*}(v)=\top$, all v
  - $(f \sqcup_{CP^*} g)(v) = f(v) \sqcup g(v)$, all v

CP is not distributive

- The CP-lattice and CP* are not distributive
  - $(1 \sqcup 2) \sqcap 3 = \top \sqcap 3 = 3 \neq$
  - $(1 \sqcap 3) \sqcup (2 \sqcap 3) = \bot \sqcup \bot = \bot$
- In practical terms, this means that the order in which the nodes are visited can affect the precision of the results

Not always finite chains

- Sometimes we will want to uses lattices that have infinite ascending chains
- In order to use such lattices in an iterative dataflow analysis, we will have to ensure that the flow functions still have only finite effective chains
- Standard example: range analysis

Range analysis

- With range analysis, we try to determine the possible range of integer values of a variable
  - elts: $\bot$, $\top$, and $[m,n]$ where $m \leq n$
  - $[m,n] \sqcup [m',n'] = [\min(m,m'), \max(n,n')]$
- A forward analysis with $\sqcup$ as the meet operator
  - How to deal with the infinite chains?

MOP is not always "best"

* For RD, LV, and AE, the flow functions are simple in the sense of being constant-time (or actually O(N), for N-variable programs)
* But the flow functions can, in principle, be arbitrarily complicated, even involving theorem proving
  * e.g.: v = x mod y, for prime x...

How fast?

* Consider reaching definitions
  * N nodes, so N possible definitions
    * N^2 bits
  * in worst case, each iteration changes one bit
    * Thus, $O(N^3)$, assuming constant-time flow functions (per bit)
* Ex: Construct a worst-case example

Why do basic blocks help?

* In terms of worst-case complexity, basic blocks do not make a difference
* However, we usually assure proper directionality within blocks, effectively assuring that the dataflow works in linear time

Reduction Methods

Iterative dataflow analysis is easy

- Iterative dataflow analysis is relatively easy to implement
- Hence, its popularity in optimizing compilers
- But it can be slow to execute
- With the rise in popularity of Just-In-Time compilers, dataflow analysis methods based on reducibility of flow graphs have re-gained interest

Reduction methods

- In reduction methods, we calculate, symbolically, a flow function for the entire procedure body
  - usually this can be done in linear time
- This is called a "reduction" because, in essence, the flow graph is reduced to a single node
- Then, in practice, a fixed point can be found in a small number (often just one) iterations

Next time...

- Overview of reduction methods
- Start on loop optimizations