Lecture 13
Abstraction in Z

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Models of Software Systems
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This Lecture

- **Establishing properties about a model**
  - > **Properties of the state space**
    - » **e.g., initial state space is not empty**
  - > **Properties of operations**
    - » **values of unspecified variables**
    - » **preconditions of operations**
- **Relating models: refinement**
  - > **State**
    - » **Abstraction relations**
    - » **Initial states property**
  - > **Operations**
    - » **Applicability**
    - » **Correctness**

You Should Know

- How to state the Initialization Theorem.
- How to apply the 1-point rule.
- How to write down the formula for calculating the pre-condition of an operation.
- How to write down an abstraction relation as a schema.
- What refinement means
  - whether one relation refines another
  - what you need to prove for schemas

Informal versus Formal Reasoning

- **Reasoning is used to increase our confidence that we have defined an appropriate model.**
  - Does my model have the desired properties?
- **Thus far (in our treatment of Z) our reasoning process has been largely informal.**
  - Not clear what we can assume when we prove something.
  - Not clear what is worth proving.
- **We will now look at a (somewhat) more formal approach.**

Formal Proofs using Z

The general form of a proof in any mathematical system is:

Hypotheses |- Conclusions

In Z the Hypotheses typically consist of:

- **Declarations -- which introduce variables**
- **Predicates - which constrain the values of of those variables**

The Conclusion is a *Predicate* that must be shown to follow as a consequence of the hypotheses.

Schemas as Hypotheses

Since a schema introduces variables and predicates, we may use them in our hypothesis:

$$\text{BirthdayBook} \vdash (\text{dom birthday}) \setminus \text{known} = \varnothing$$

$$\text{AddBirthday} \vdash \text{known}' = \text{known} \cup \{\text{name?}\}$$

Global (axiomatic) declarations are implicitly included:

$$\vdash \text{maxsize}: \mathbb{N}$$

$$\text{AddBirthday}; \#\text{known} < \text{maxsize} \vdash \#\text{known}' \leq \text{maxsize}$$

Schemas as Hypotheses (cont)

So, a theorem of the form:

BirthdayBook |- (dom birthday) \ known = ∅

actually means:

known: ℙ NAME; birthday: NAME → DATE

| known = *dom* birthday

|-

(dom birthday) \ known = ∅

To prove this we would argue as follows:

(dom birthday) \ known

= known \ known **[hypothesis]**

= ∅ **[property of \ ]**

Schemas as Conclusions

Formally, a conclusion must be a predicate.

We can use a schema in the place of a conclusion

- **The predicate of the schema becomes the conclusion**
- **Provided that the variables are all within scope**

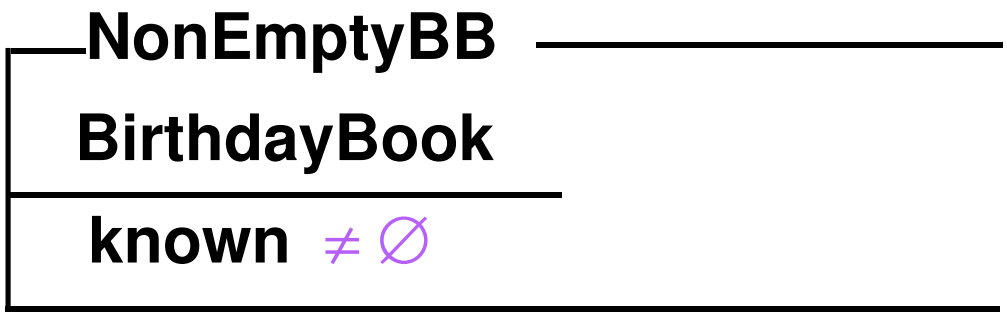
Examples

InitTN |- EfficientTN

name? ∉ known; (AddBirthday ; DeleteBirthday)

|- Ξ BirthdayBook

Example



AddBirthday |- NonEmptyBB' ≡

known, known': ℙ NAME;
birthday, birthday': NAME → DATE;
name?: NAME; date?: DATE
| known = *dom* birthday ∧ known' = *dom* birthday'
name? ∉ known
birthday' = birthday ∪ { (name?, date?) }
|-
known' ≠ ∅

The Initialization Theorem

Given a description of the initial state of a system, we would like to guarantee that at least one such state exists.

Suppose InitState is a schema describing the initial state of a system whose state is defined by schema State

Formally, we need to prove:

$\vdash \exists$ **State' • InitState**

(note in this case we have no hypotheses)

Example

InitBirthdayBook

BirthdayBook'

---

known' = ∅

birthday' = ∅

We mustshow

|- ∃ BirthdayBook' • InitBirthBook

Example (cont)

$\vdash \exists$ BirthdayBook' • InitBirthBook

$\equiv$

$\vdash \exists$ known': $\mathbb{P}$ NAME; birthday': NAME $\rightarrow$DATE

| known' = *dom* birthday'

• known' = $\varnothing \wedge$ birthday' = $\varnothing$

Informally we might argue this as follows:

- Take known' and birthday' to be $\varnothing$.
- For these choices the state invariant known' = *dom* birthday' is satisified.
- This demonstates the existence of an appropriate known' and birthday'.

The One-Point Rule

In a more formal proof we might prove the theorem by simplifying the above to: |- true.

To do this we would have to eliminate the $\exists$.

A key tool for doing this is the <u>One Point Rule</u>

$$\frac{\exists\ x: S \bullet (p \wedge x = t)}{p[t/x]} \qquad (x \text{ is not free in } t)$$

i.e., if you know the value of a variable, you can substitute the value for the variable and eliminate it from the existential quantification.

Two Useful Identities

1. $\forall$ x: T | P(x) • Q(x)

$\equiv$

$\forall$ x: T • P(x) $\Rightarrow$ Q(x)

2. $\exists$ x: T | P(x) • Q(x)

$\equiv$

$\exists$ x: T • P(x) $\wedge$ Q(x)

Example

$\exists$ **known': $\mathbb{P}$ NAME; birthday': NAME $\rightarrow$ DATE**

| known' = *dom* birthday'

• known' = $\varnothing$ $\wedge$ birthday' = $\varnothing$

$\equiv$ [logic]

$\exists$ **known': $\mathbb{P}$ NAME; birthday': NAME $\rightarrow$ DATE**

• known' = *dom* birthday' $\wedge$ known' = $\varnothing$ $\wedge$ birthday' = $\varnothing$

$\equiv$ [1 point rule -- known' = $\varnothing$]

$\exists$ **birthday': NAME $\rightarrow$ DATE**

• $\varnothing$ = *dom* birthday' $\wedge$ birthday' = $\varnothing$

$\equiv$ [1 point rule -- birthday' = $\varnothing$]

($\varnothing$ = dom $\varnothing$)

$\equiv$ [property of dom]

true

Precondition Calculation

- When an operation is defined it is important to know in which states it can be applied.
- Other cases can then be identified and error messages supplied.
- The "precondition" tells us the starting states from which the operation could be successfully applied
- Informally the precondition is obtained by looking at the part of the operation's predicate that contains only inputs and "before" variables.

Example

AddBirthday

Δ **BirthdayBook**

name?: NAME

date?: DATE

$$name? \notin known$$

$$birthday' = birthday \cup \{ name? \mapsto date? \}$$

has precondition: $name? \notin known$

Preconditions (cont)

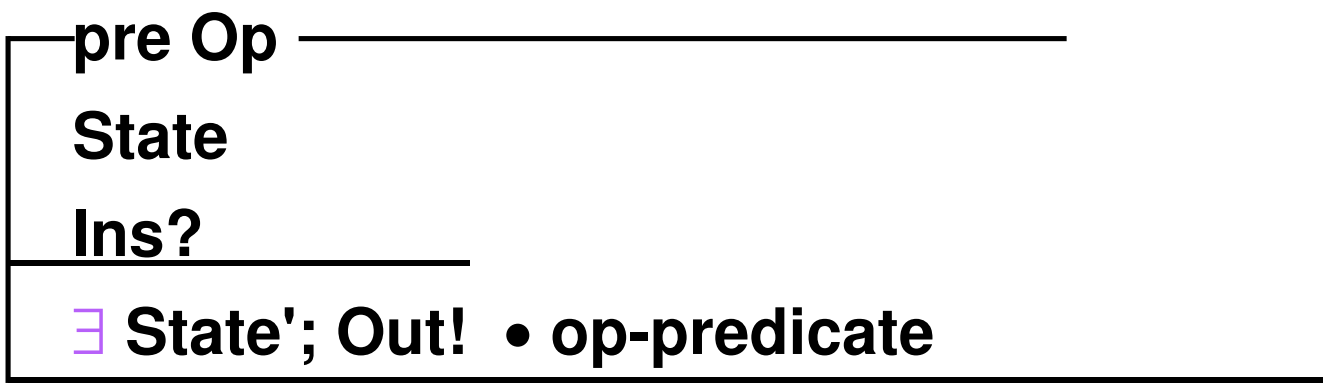
More formally we would like a rule for determining the precondition of a schema.

We do this by "hiding" the primed and output variables.

If Op is an operation on state State with outputs Outs!, then we define pre Op (the precondition of Op) to be the schema:

$\exists$ State'; Out! • Op

Formally this is the schema



Example

pre AddBirthday

BirthdayBook

name?: NAME

date?: DATE

$$\exists\ \mathbf{BirthdayBook'} \bullet$$

$$\mathbf{name?} \notin \mathbf{known} \land$$

$$\mathbf{birthday'} = \mathbf{birthday} \cup \{ (\mathbf{name?}, \mathbf{date?}) \}$$

Example (cont)

$\exists$ **known': $\mathbb{P}$ NAME; birthday': NAME $\rightarrow$ DATE**

- **known' = *dom* birthday' $\wedge$ name? $\notin$ known $\wedge$**
  birthday' = birthday $\cup$ {(name?, date?)}

$\equiv$ [1 point rule -- known']

$\exists$ **birthday': NAME $\rightarrow$ DATE**

- **name? $\notin$ known $\wedge$**
  birthday' = birthday $\cup$ {(name?, date?)}

$\equiv$ [1 point rule -- birthday']

name? $\notin$ known

General Reasoning

Other useful things can be proved, but these typically depend on the application.

Example

name? $\notin$ known; AddBirthday ; DeleteBirthday

- **Ξ BirthdayBook**

(question: why is the first hypothesis needed?)

Implementation/Refinement

- Qn: Given two state machines, what does it mean to say one implements another?
- Ans: The second machine should do at least what the first machine does.
- Note: It can do more, and if the abstract machine leaves some options open, then it can chose one of those options.

Methods of Refinement in Z

- **Basic problem: How get from a specification to a program?**
- **No single method**
- **Usually development is divided into two parts**

  Part1:

  repeat two step process of

  – data refinement

  – operation refinement

  until "sufficiently close" to a programming language

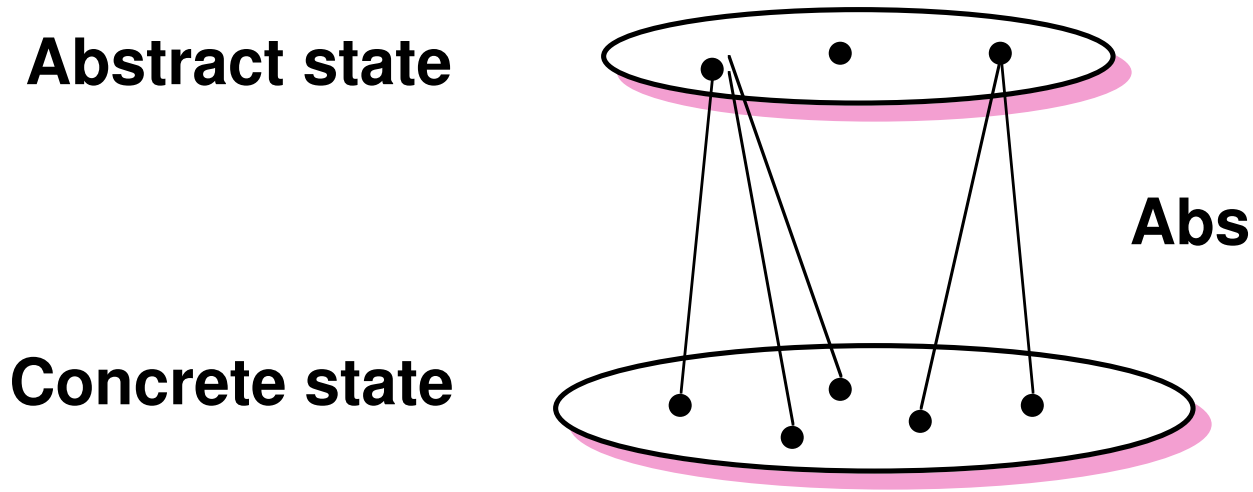
  Part 2:

  "associate" code with the specification

Relating Two State Machines

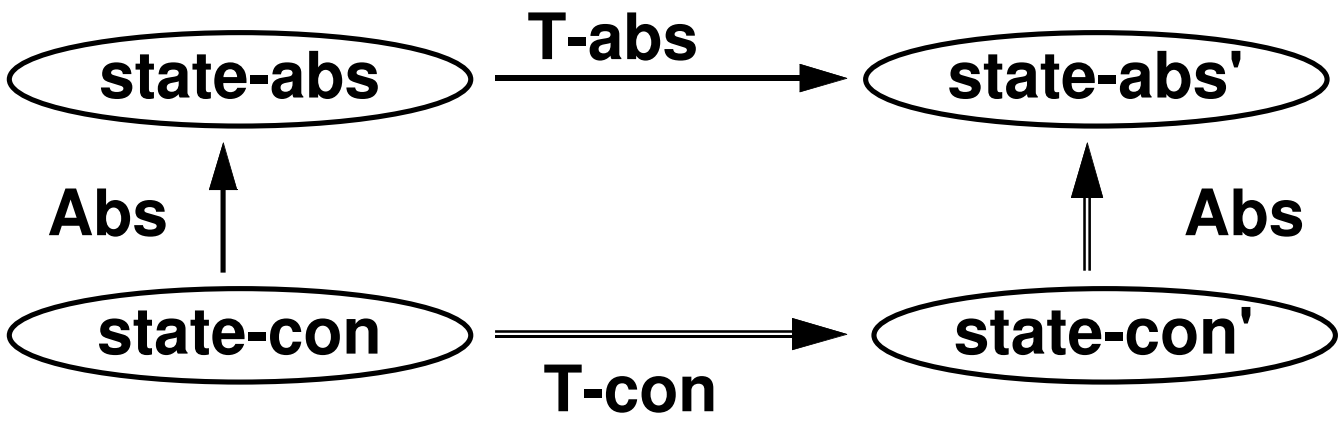
In general, two machines may have different state spaces :

Our first task is therefore to produce an abstraction relation from the "concrete" machine to the "abstract" machine (Abs).



Then for each state transition, T, establish the correspondence.

T-abs (Abs(state-con)) = Abs(T-con(state-con)



Observations

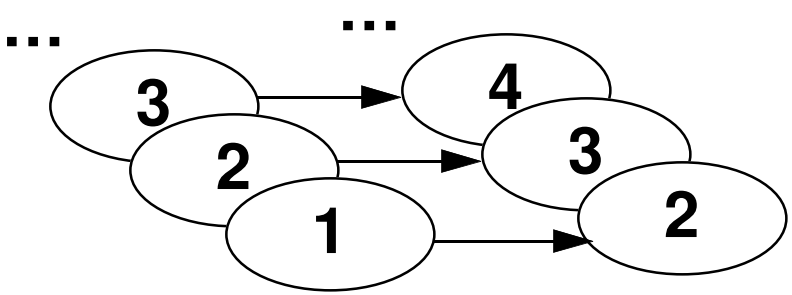
- **Abs usually associates many concrete states with each abstract state.**

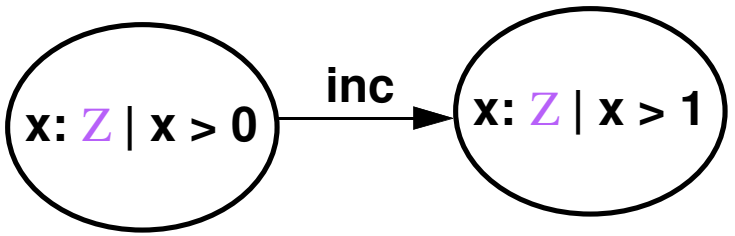
  (eg., set represented by a sequence)
- **Abs is usually a function: each concrete state has at most one abstract state.**

  (If not, what does this say about the abstract state model?)
- **Abs is sometimes called a "retrieve" function.**
- **Must also account for initial states.**
- **Concrete operations**
  - must be defined for all the corresponding states for which the abstract operation is defined (applicability)
  - must produce appropriate values (correctness)

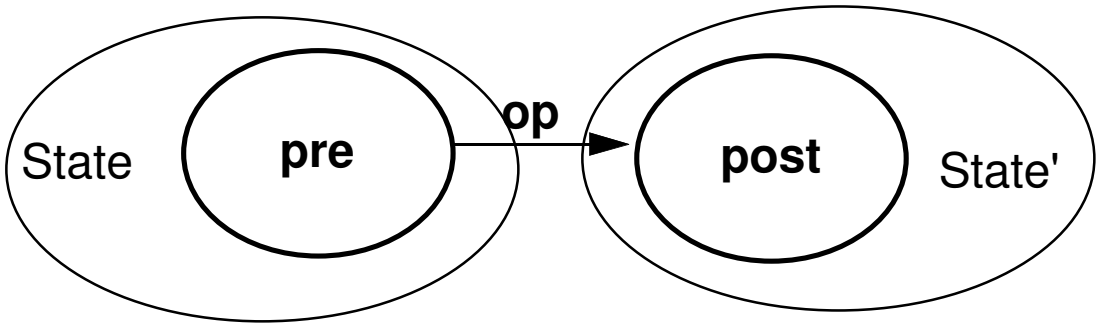
Recall

- For state machines with large state spaces diagramatic techniques are inadequate.
- Hence, we often describe a *collection* of states by indicating some property that they all have.
- Many transitions on many states can be collapsed into a single (relational) description in this way.
- Example:





- **In Z we first describe the global domain of discourse as a set of states (defined by a schema) – the state space.**
- **Then we define the operations in terms of the transition relationships between subsets of these states.**
- **For each operation:**
  - **The pre-condition is a predicate that determines the legal starting states.**
  - **The post-condition is a predicate that determines the resulting states.**



Data Refinement in Z

BirthdayBook

known: $\mathbb{P}$ NAME

birthday: NAME $\rightarrow$ DATE

---

known = *dom* birthday

BirthdayBook1

names: $\mathbb{N}_1 \rightarrow$ NAME

dates: $\mathbb{N}_1 \rightarrow$ DATE

size: $\mathbb{N}$

array[1..] of ...

$\forall$ i, j: 1 .. size • i $\neq$ j $\Rightarrow$ names(i) $\neq$ names(j)

Abstraction Function

Abs

BirthdayBook

BirthdayBook1

---

known = {i: 1.. size • names(i)}

∀ i, j: 1 .. size • birthday(names(i)) = dates(i))

The Initial States Theorem

- **Must show that every initial concrete state corresponds to an initial abstract state.**
- **This is the theorem:**

$$\text{InitConcrete} \wedge \text{Abs}' \Rightarrow \text{InitAbstract}$$

- **Example:**

InitBirthBook

BirthdayBook'

known' = $\varnothing$

birthday' = $\varnothing$

InitBirthBook1

BirthdayBook1'

size' = 0

names' = dates' = <>

To prove

InitConcrete ∧ Abs' |- InitAbstract

we would argue:

known

= {i: 1.. size • names(i)} [Abs']

= { i: 1.. 0 • names(i)} [size = 0]

= ∅

and similarly we could show

birthday = ∅

Operation Refinement

- When a concrete operation Op2 refines an abstract operation Op1, we write: Op1 Op2
- To show that Op2 is a refinement of Op2:

  1. Applicability

     Whenever Op1 is applicable so is Op2.

  2. Correctness

     When Op1 is applicable but Op2 is applied, then the result is consistent with Op1.

Relations

- **For the case of relations R and S, R S means:**
  1. **dom R † dom S** [applicability]
  2. **((dom R) ‘ S) † R** [correctness]
- **Example**

  R = {Todd ¢ blue, Todd ¢ red, Jim ¢ green, Jim ¢ red}

  S = {Todd ¢ blue, Jim ¢ green, Hillary ¢ red}

R S because

1. **{Todd, Jim} † {Todd, Jim, Hillary}**
2. **{Todd ¢ blue, Jim ¢ green} † R**

Schemas

- **For the case of schemas, R and S, that represent operations on the same state, R S means:**

  1. pre R pre S [applicability]

  2. (pre R $\wedge$ S) R [correctness]

General Case

- **In the general case, the operations apply to different states (concrete and abstract).**
- **In this case Op-abs Op-conc means:**

  1. pre Op-abs $\wedge$ Abs pre Op-conc [applicability]

  2. (pre Op-abs $\wedge$ Abs $\wedge$ Op-conc)

  $\exists$ **State' • Op-abs $\wedge$ Abs'** [correctness]

- **Sometimes 2 is written**

  pre Op-abs $\wedge$ Abs $\wedge$ Op-conc $\wedge$ Abs'

  Op-abs

Example

AddBirthday

Δ **BirthdayBook**

name?: NAME

date?: DATE

$$name? \notin known$$

$$birthday' = birthday \cup \{ name? \mapsto date? \}$$

AddBirthday1

Δ BirthdayBook1

name?: NAME

date?: DATE

$\forall$ i: 1 .. size • names? $\neq$ names(i)

size + 1 = size'

names' = names $\oplus$ {size' $\mapsto$ name?}

dates' = dates $\oplus$ {size $\mapsto$ date?}

Applicability (informal)

- **Must show that whenever AddBirthday can be legally applied to an abstract state, AddBirthday1 can be legally applied to any corresponding concrete state.**
- **The precondition of AddBirthday1 is**

  $\forall$ **i: 1 .. size • name?** $\neq$ **names(i)**

- **Abs tells us that this precondition is equivalent to saying**

  name? $\notin$ **known**

- **But this is precisely the precondition of AddBirthday**

Correctness (informal)

- We must show that the concrete operation is consistent with the postcondition of AddBirthday:

  birthday' = birthday $\cup$ { name? ¢ date? }

- The proof goes as follows:

  1. We show that the function represented by the two arrays after the operation has the same domain as birthday' (as viewed through Abs).

  2. Then we show that the ranges are the same.