Solving Linear Systems (on parallel Processors)

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What are Linear Systems?

$$A\mathbf{x} = \mathbf{b}$$

What are Linear Systems?

$$
\begin{bmatrix}
a_{11} & a_{12} & a_{13} & a_{14} \\
a_{21} & a_{22} & a_{23} & a_{24} \\
a_{31} & a_{32} & a_{33} & a_{34} \\
a_{41} & a_{42} & a_{43} & a_{44}
\end{bmatrix}
\begin{bmatrix}
x_1 \\ x_2 \\ x_3 \\ x_4
\end{bmatrix}
=
\begin{bmatrix}
b_1 \\ b_2 \\ b_3 \\ b_4
\end{bmatrix}
$$

Why Do We Care?

Inverse Problems:

$$\mathbf{f}(\mathbf{x}) = \mathbf{y} \qquad \leftarrow \text{solve}$$

$$\mathbf{f}(\mathbf{x}) \approx \mathbf{f}(\mathbf{x}_0) + \nabla \mathbf{f}(\mathbf{x}_0)\big(\mathbf{x} - \mathbf{x}_0\big)$$

$$\mathbf{y} \approx \big(\nabla \mathbf{f}(\mathbf{x}_0)\big)^{-1}\big(\mathbf{y} - \mathbf{f}(\mathbf{x}_0)\big)$$

Optimization Problems:

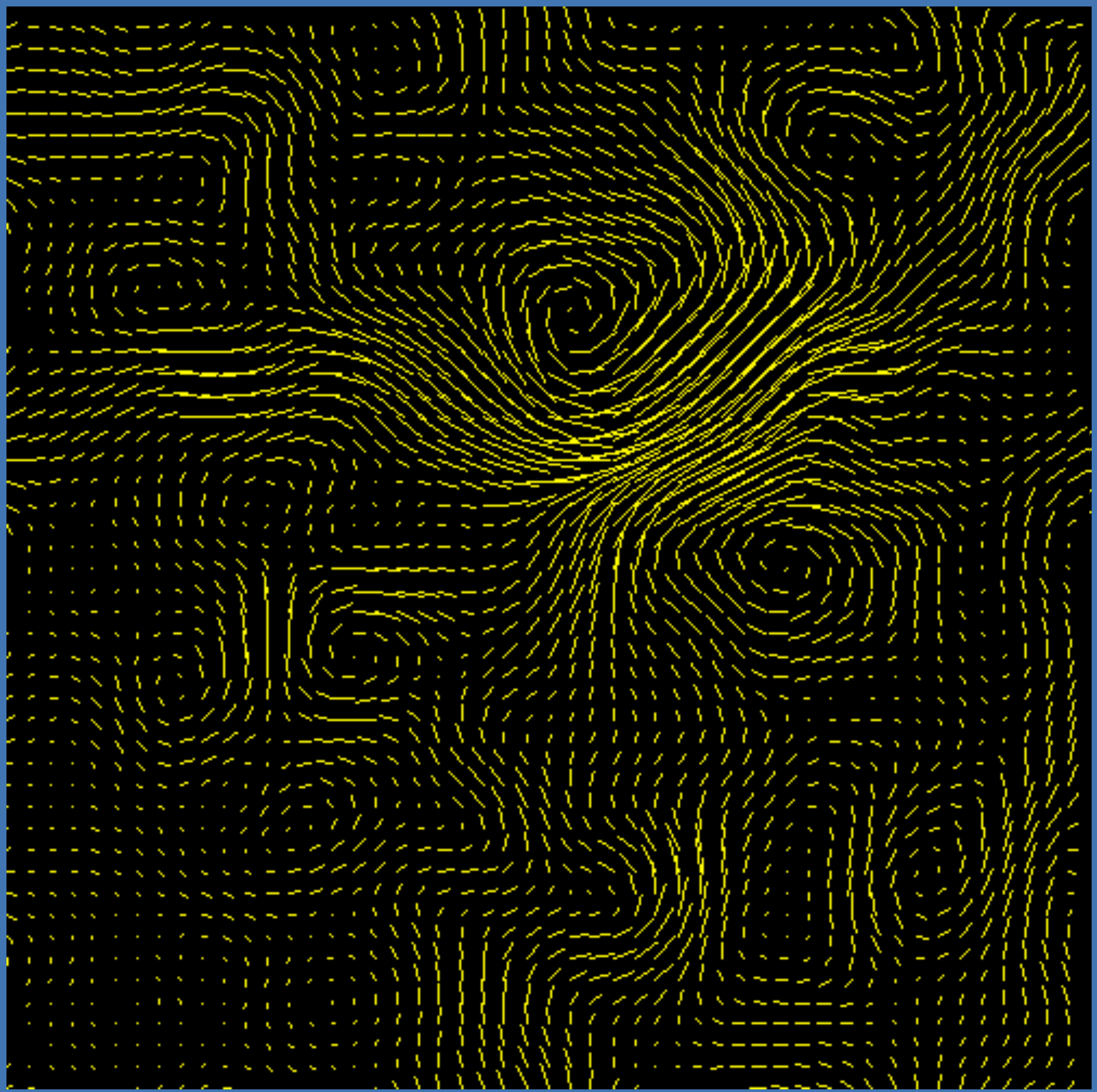
$$\mathbf{f}(\mathbf{x}) \qquad \leftarrow \text{minimize}$$

$$\nabla \mathbf{f}(\mathbf{x}) = \mathbf{0}$$

$$M\mathbf{x} = \mathbf{b} \qquad \text{(if } f \text{ is quadratic)}$$

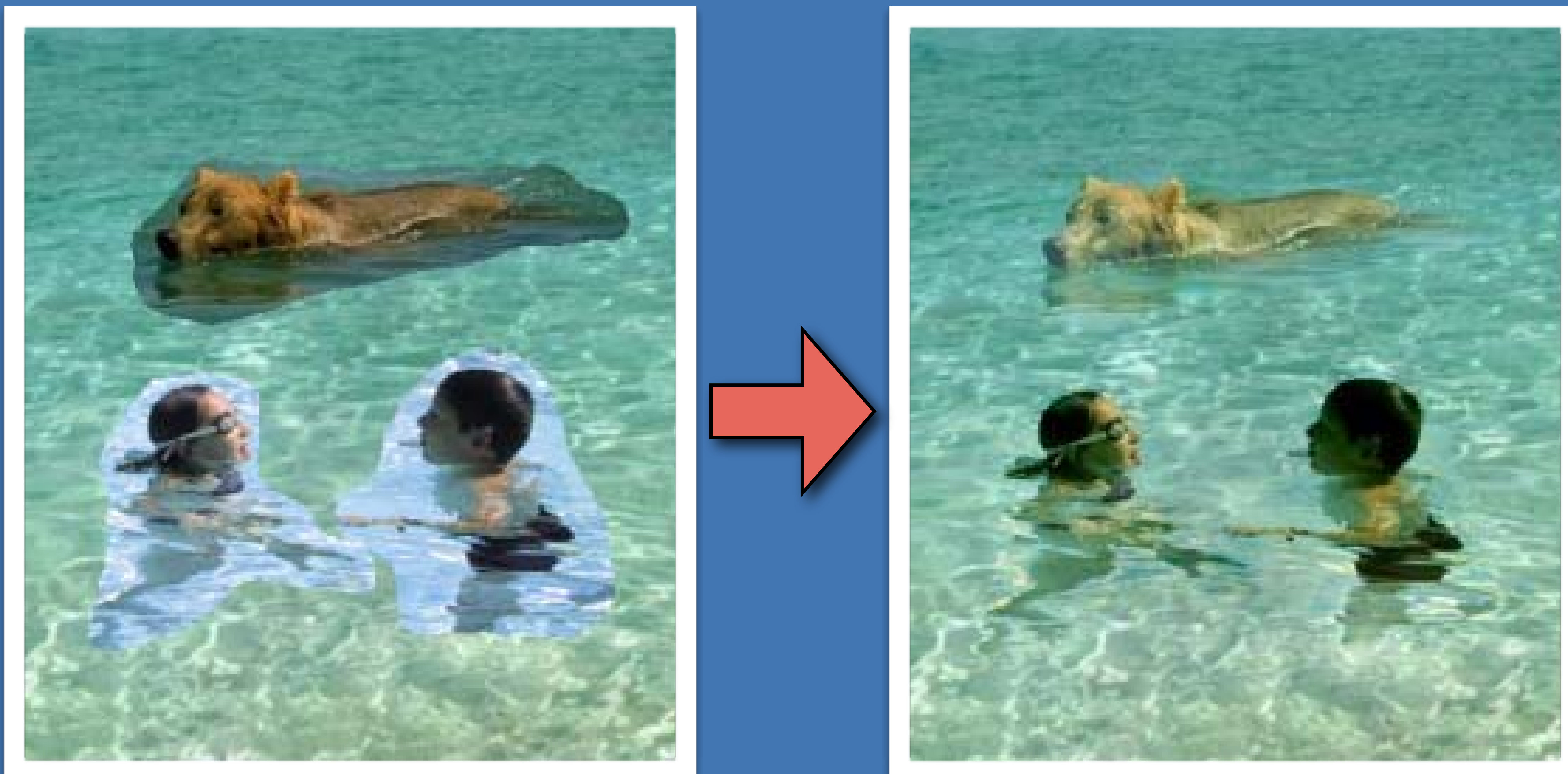
Fluid Dynamics

$$P\left(\right.$$

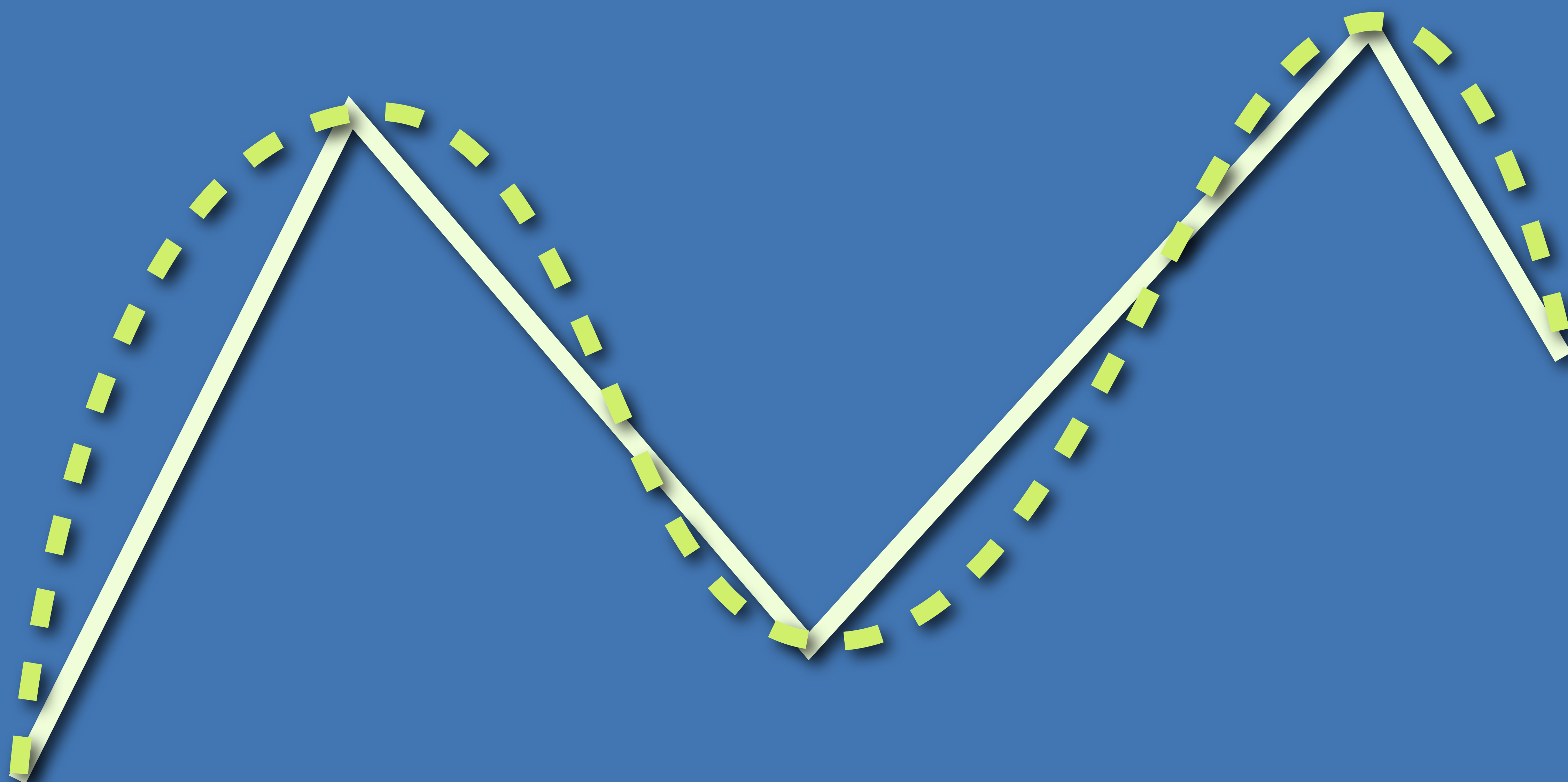


$$\left. \right)$$

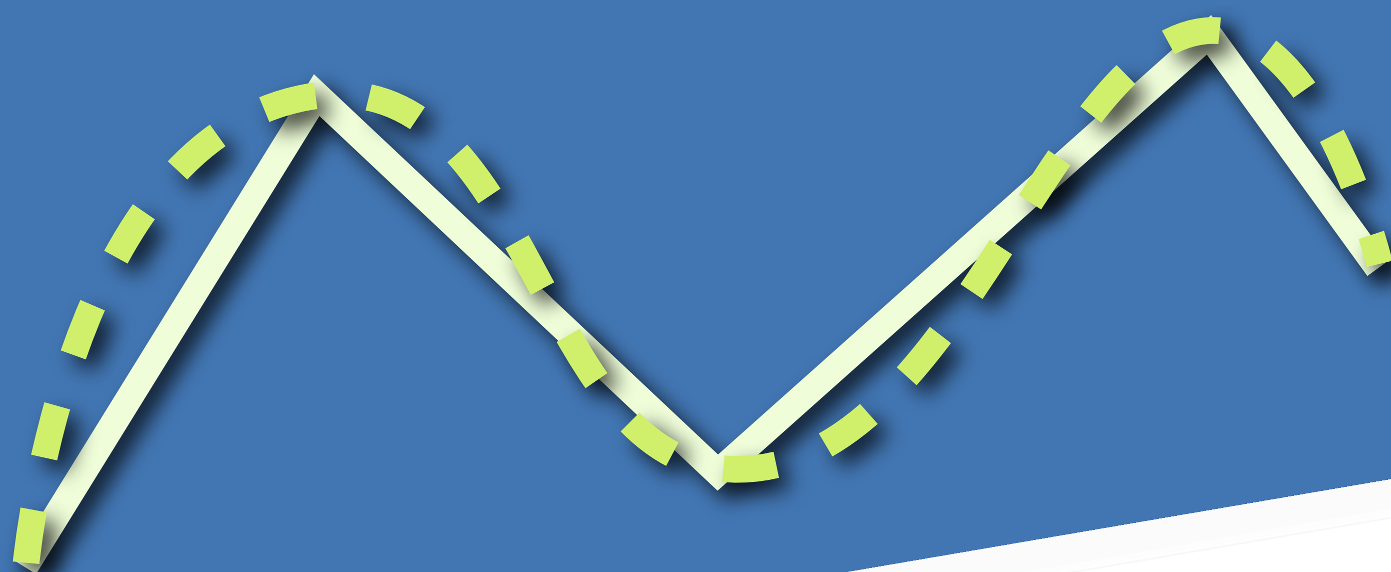
Poisson Blending



Rasterizing Curves



QUESTION!!!



$$A\mathbf{x} = \mathbf{b}$$

How could we solve such systems? (Efficiently on the GPU?)

Gauss-Seidel

Board

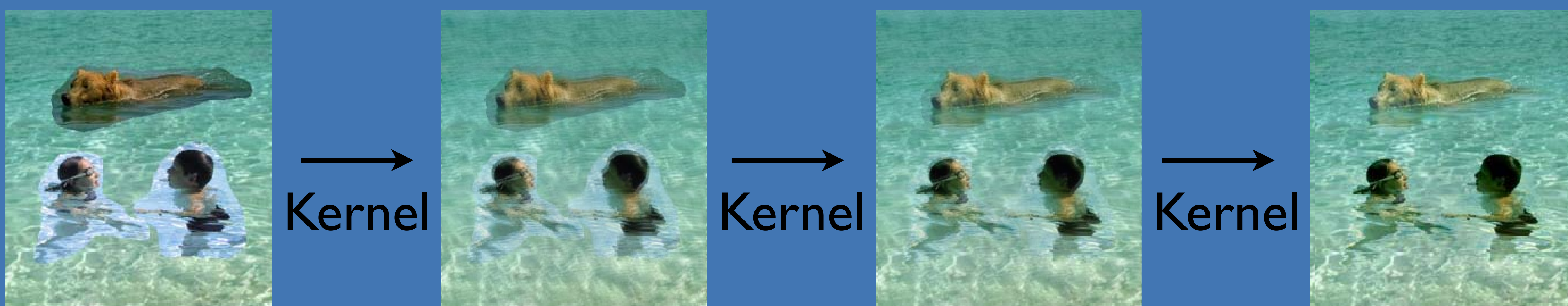
Jacobi

Board

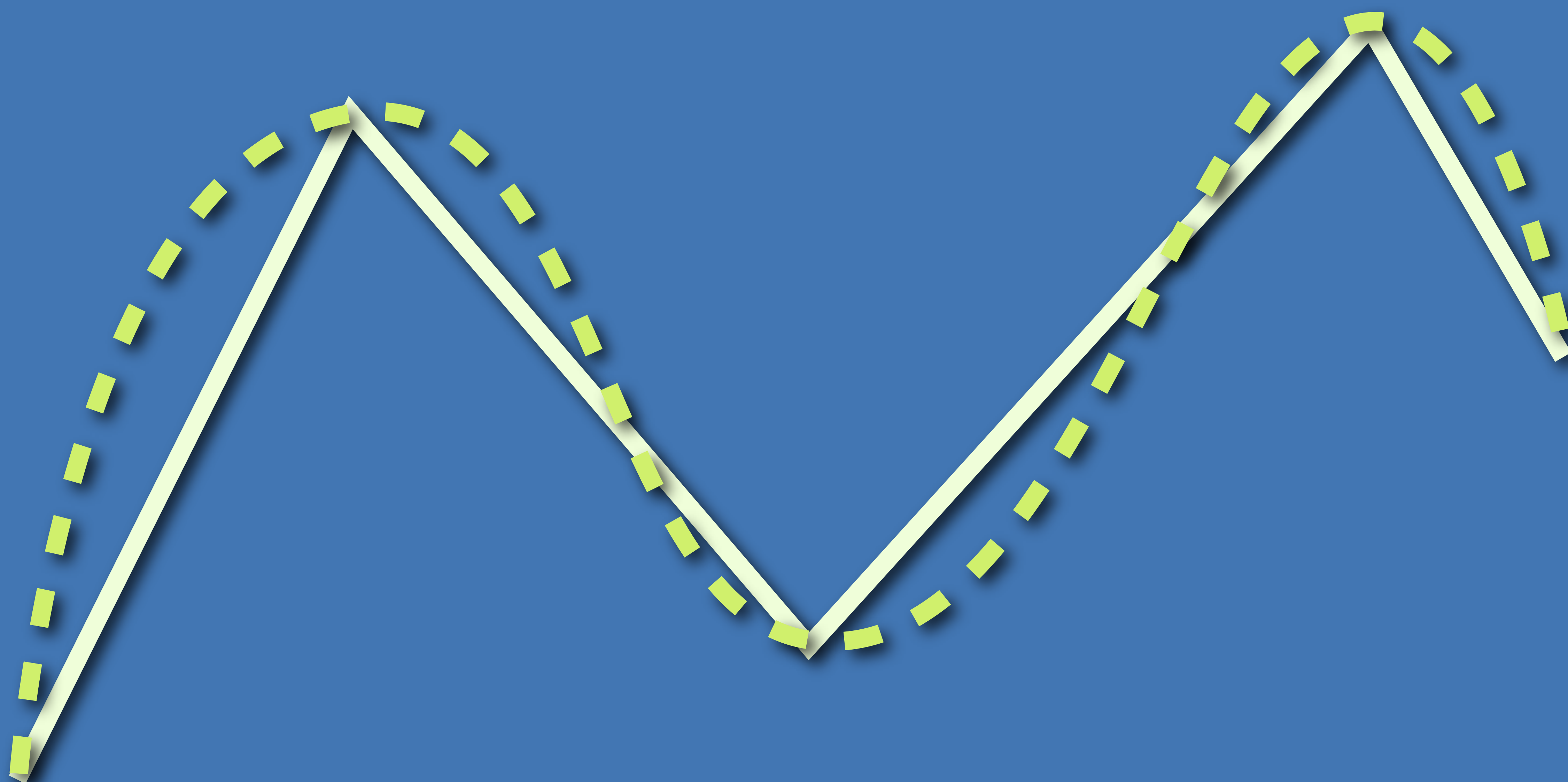
On the GPU...

$$\begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} \xrightarrow[\text{Kernel}]{} \begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} \xrightarrow[\text{Kernel}]{} \begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} \xrightarrow[\text{Kernel}]{} \begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}$$

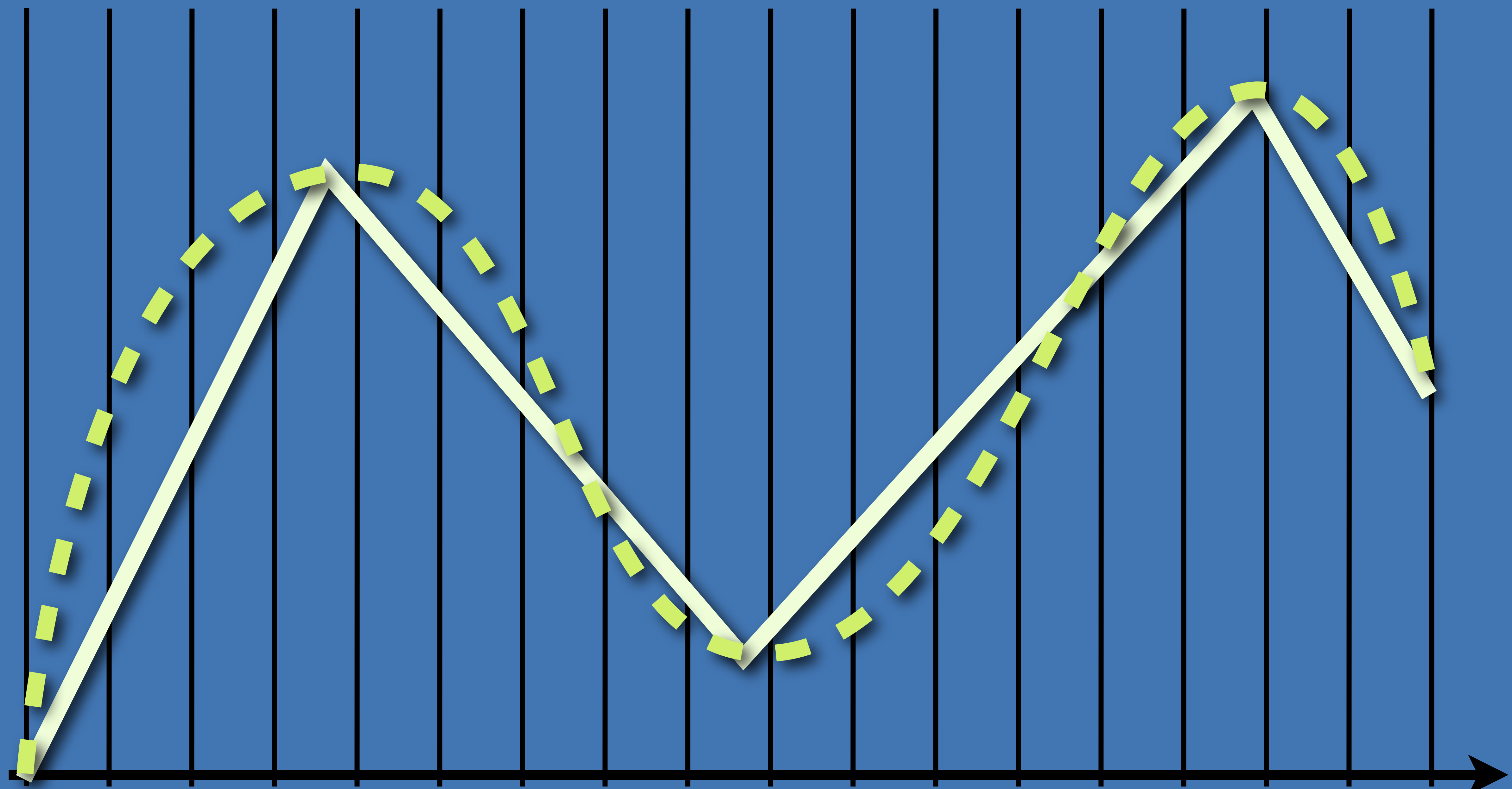
On the GPU...



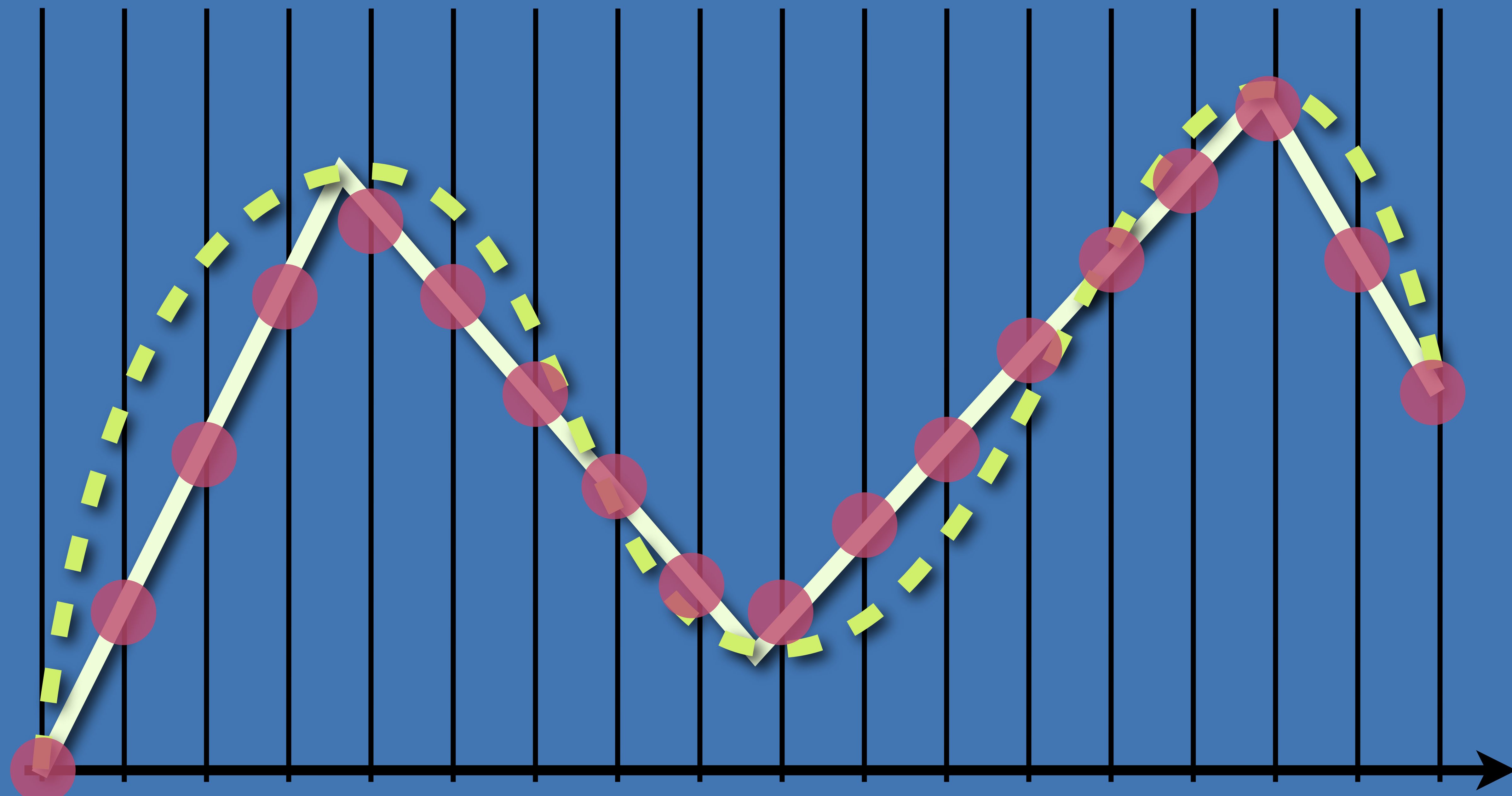
Rasterizing Curves



Example

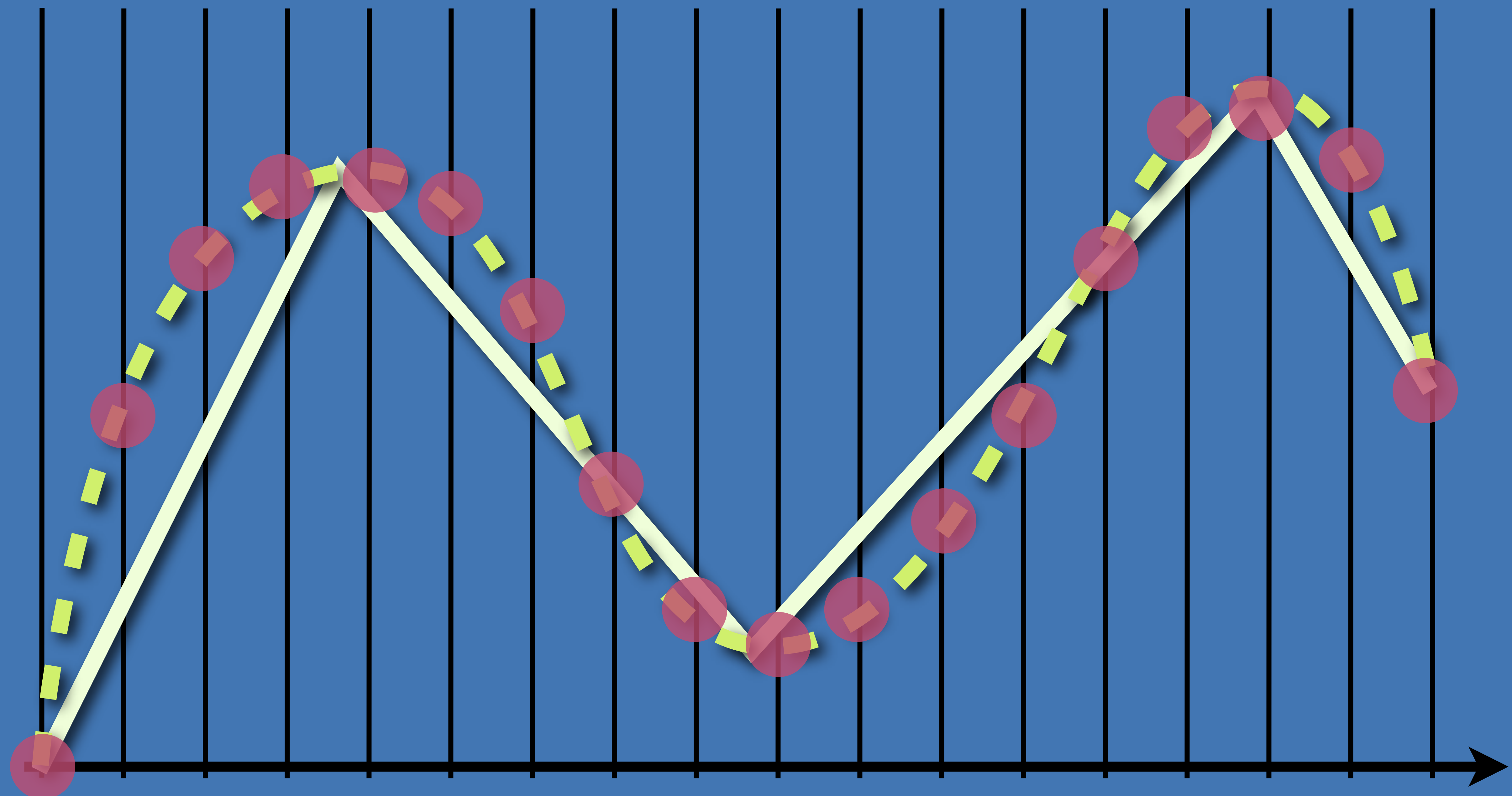


Example



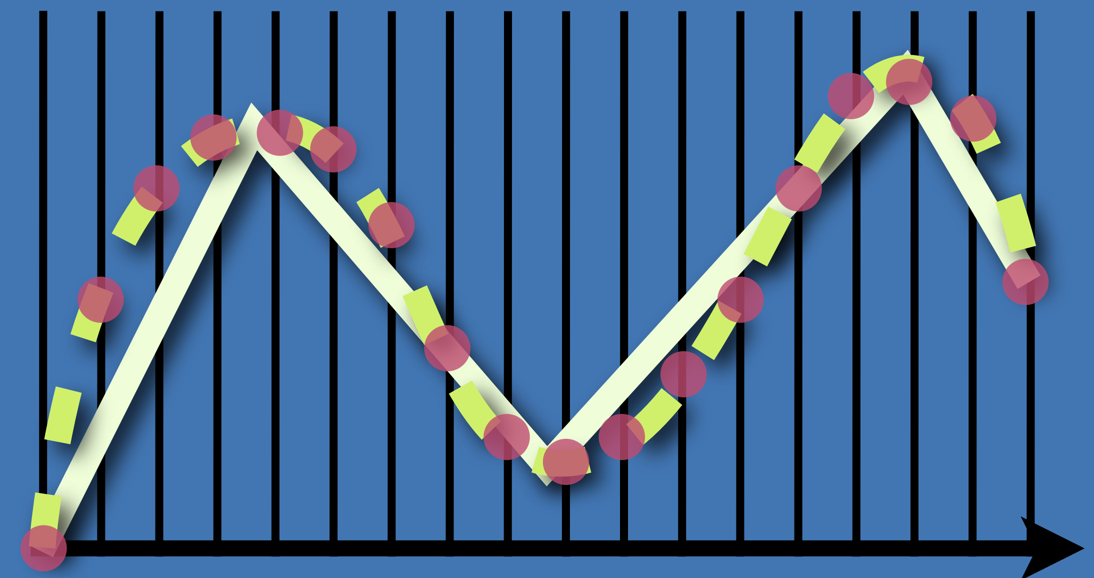
$$\mathbf{d} = [d_0, \ldots, d_N]^T$$

Example



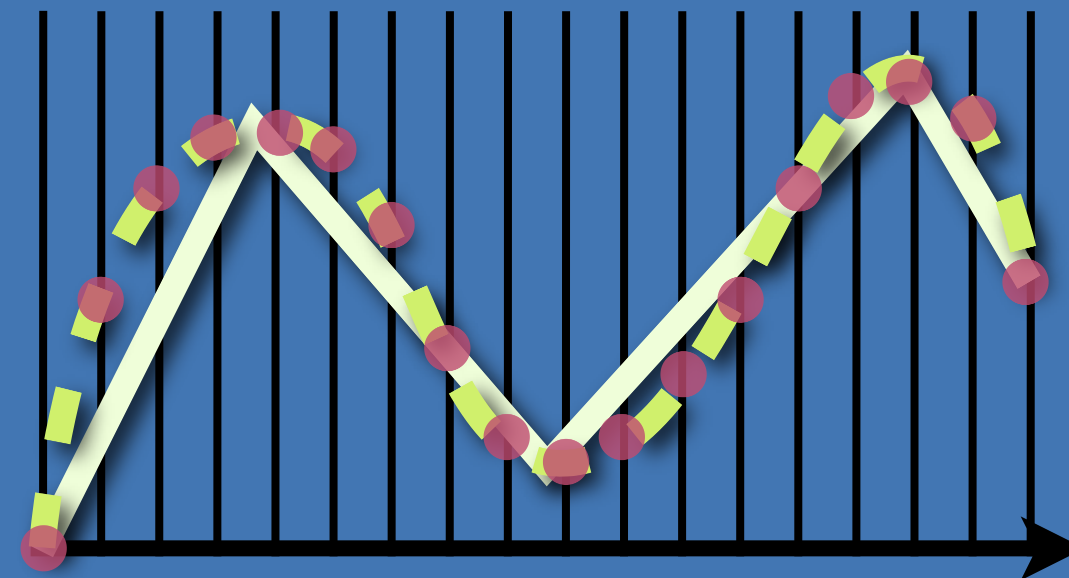
$$\mathbf{x} = [x_0, \ldots, x_N]^T$$

Example



$$\phi(x) = \alpha \times \mathrm{data}(\mathbf{x}) + \mathrm{smooth}(\mathbf{x})$$

Example

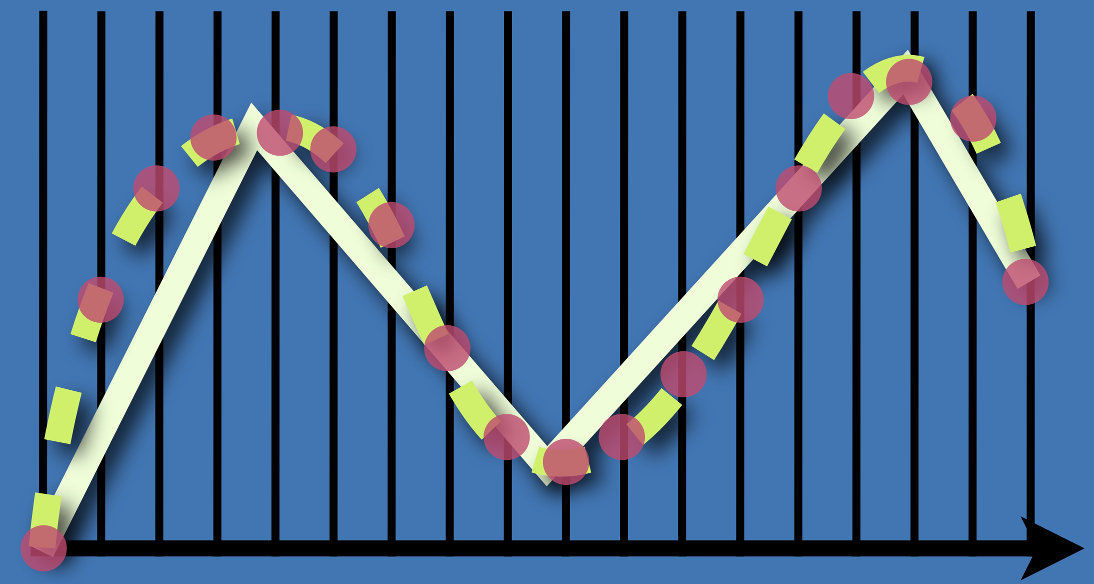


$$\phi(x) = \alpha \times \text{data}(\mathbf{x}) + \text{smooth}(\mathbf{x})$$

$$\text{data}(\mathbf{x}) = \sum_{i=1}^{N} (x_i - d_i)^2$$

$$\text{smooth}(\mathbf{x}) = \sum_{i=1}^{N-1} (x_i - x_{i+1})^2$$

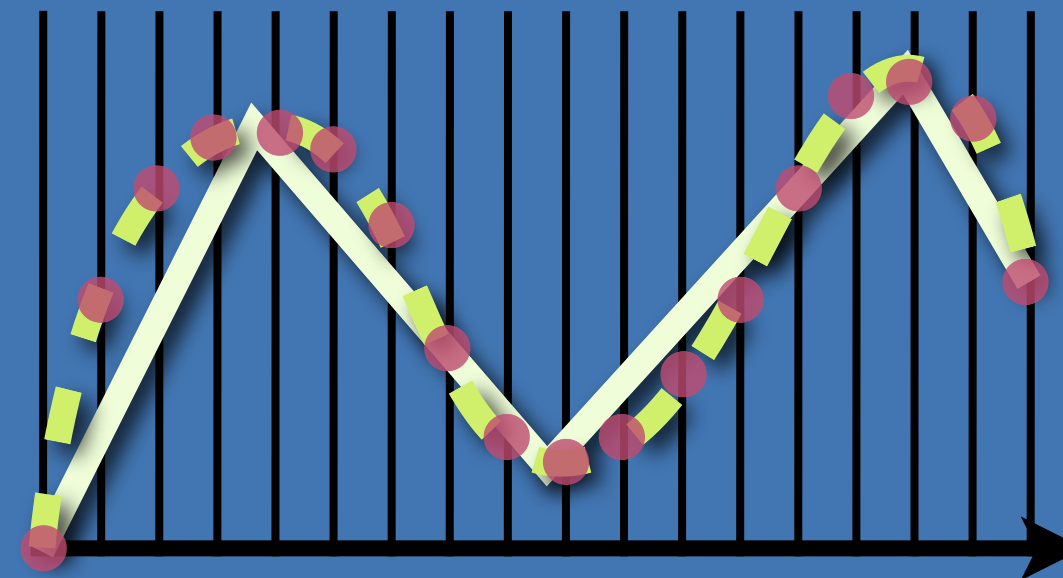
Example

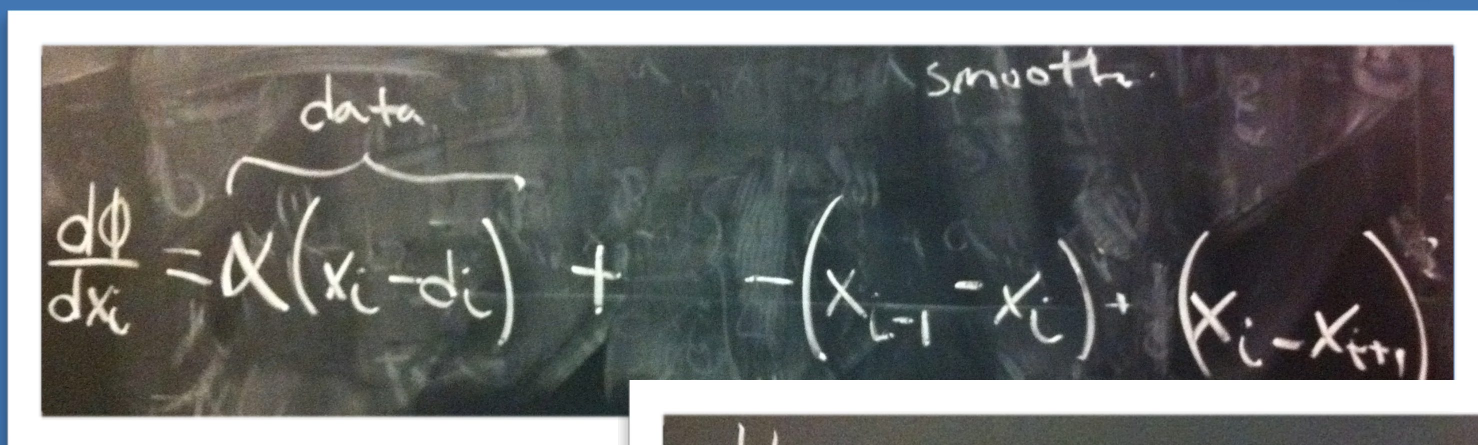


$$\phi(x) = \alpha \times \mathrm{data}(\mathbf{x}) + \mathrm{smooth}(\mathbf{x})$$

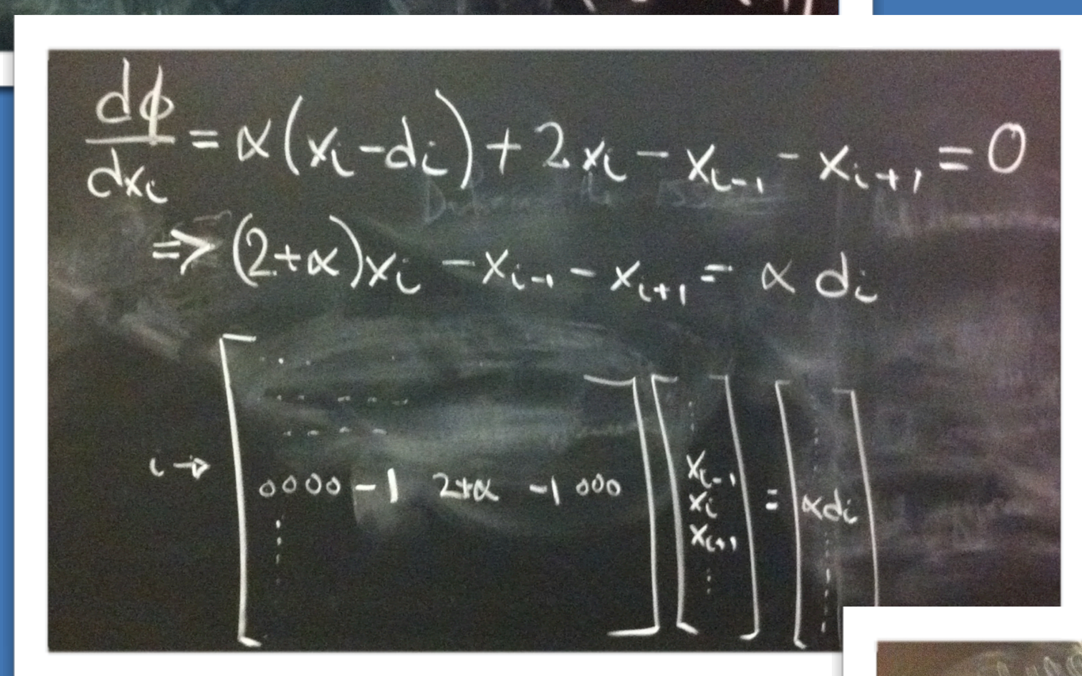
$$\frac{d\phi}{d\mathbf{x}} = \mathbf{0}$$

Example

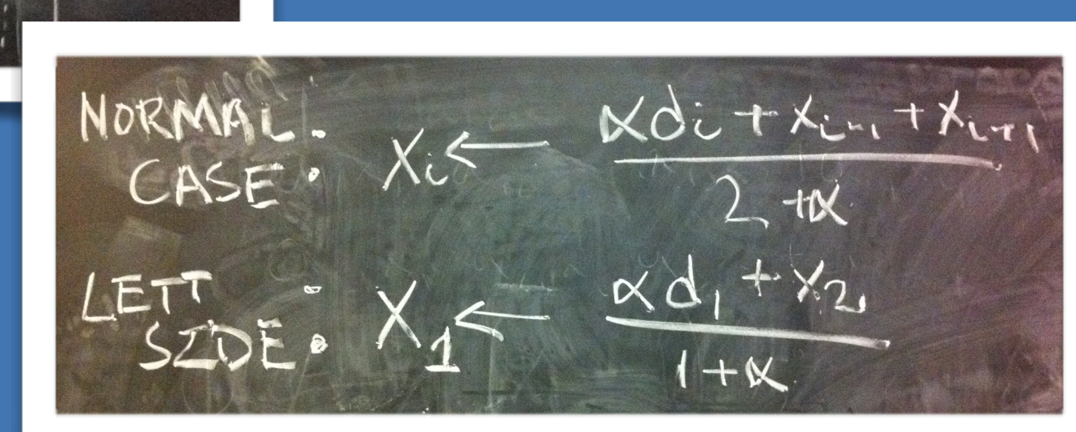




data
smooth
$\frac{d\phi}{dx_i} = \alpha(x_i - d_i) + -(x_{i-1} - x_i) + (x_i - x_{i+1})$

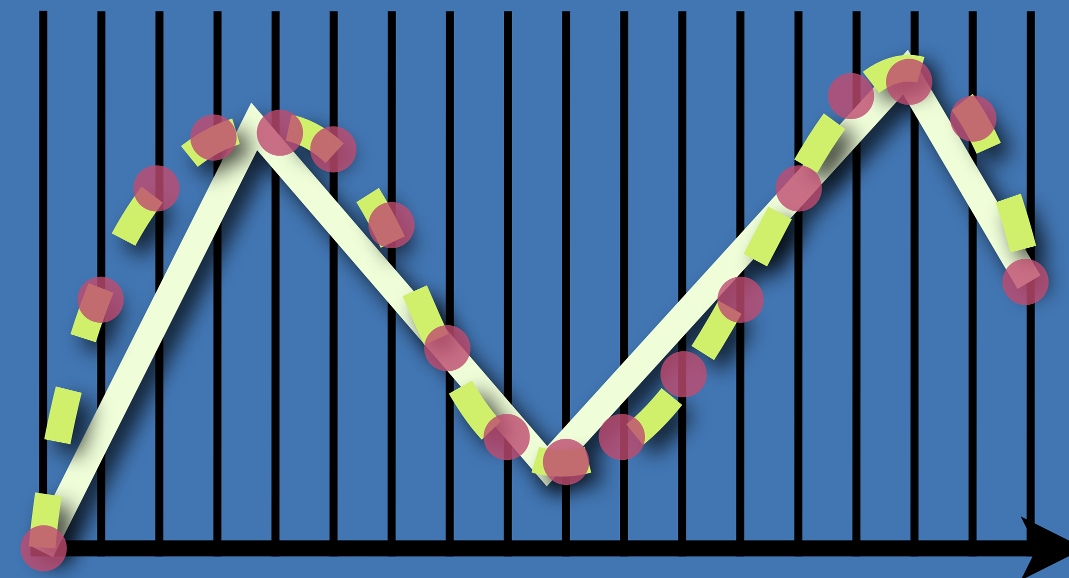


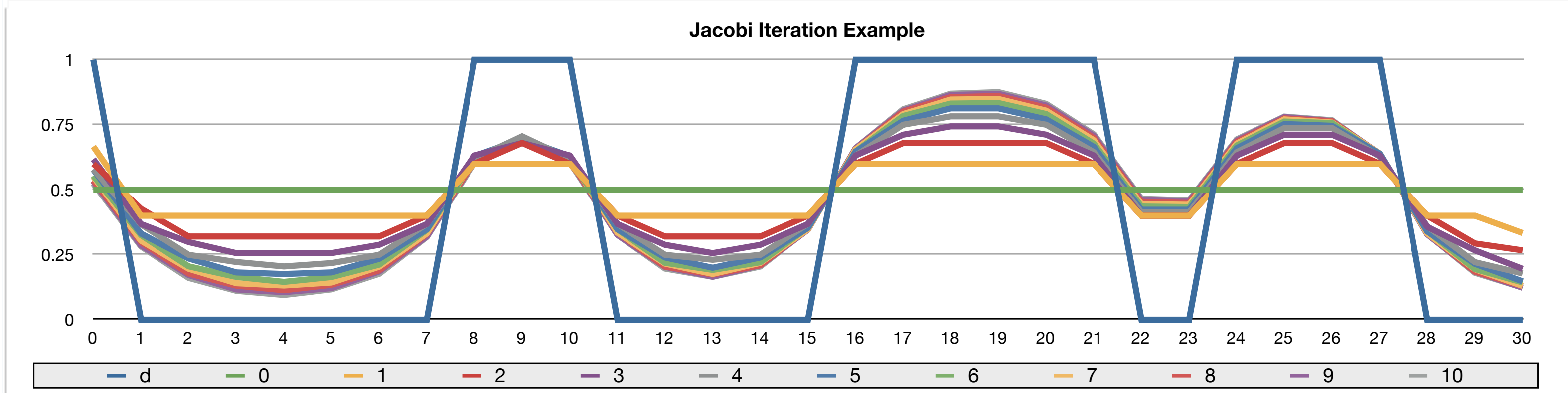
$\frac{d\phi}{dx_i} = \alpha(x_i - d_i) + 2x_i - x_{i-1} - x_{i+1} = 0$
$\Rightarrow (2+\alpha)x_i - x_{i-1} - x_{i+1} = \alpha d_i$
$i \rightarrow$
0000 −1 2+α −1 000
x_{i-1}
x_i
x_{i+1}
αd_i



NORMAL CASE: $X_i \leftarrow \frac{\alpha d_i + x_{i-1} + x_{i+1}}{2+\alpha}$
LEFT SIDE: $X_1 \leftarrow \frac{\alpha d_1 + x_2}{1+\alpha}$

Example





Alpha **Target** **Iterations→**

←Data Points

0.5	d	0	1	2	3	4	5	6	7	8	9	10
0	1	0.5	0.6666666667	0.6	0.6177777778	0.5786666667	0.5777185185	0.5542044444	0.5501590123	0.5358482963	0.5322775282	0.5235890189
1	0	0.5	0.4	0.4266666667	0.368	0.3665777778	0.3313066667	0.3252385185	0.3037724444	0.2984162923	0.2853835283	0.281603045
2	0	0.5	0.4	0.32	0.2986666667	0.2496	0.2353777778	0.2052266667	0.1958817185	0.1776105244	0.1717300843	0.1606896147
3	0	0.5	0.4	0.32	0.256	0.2218666667	0.18176	0.1644657778	0.1402538667	0.1309089185	0.1163405084	0.1110892139
4	0	0.5	0.4	0.32	0.256	0.2048	0.1757866667	0.145408	0.1313905778	0.1132407467	0.1059929505	0.0950159132
5	0	0.5	0.4	0.32	0.256	0.2176	0.18176	0.1640106667	0.142848	0.1340734578	0.1211992747	0.1165467041
6	0	0.5	0.4	0.32	0.288	0.2496	0.23424	0.211712	0.2037930667	0.18975744	0.1853738098	0.1764580011
7	0	0.5	0.4	0.4	0.368	0.368	0.34752	0.345472	0.3315456	0.3293610667	0.319945728	0.3182097522
8	1	0.5	0.6	0.6	0.632	0.6192	0.62944	0.617152	0.6196096	0.61010688	0.6101505707	0.6033959936
9	1	0.5	0.6	0.68	0.68	0.7056	0.69536	0.703552	0.6937216	0.69601536	0.688544256	0.6889287339
10	1	0.5	0.6	0.6	0.632	0.6192	0.62944	0.617152	0.6204288	0.61125376	0.612171264	0.605879808
11	0	0.5	0.4	0.4	0.368	0.368	0.34752	0.34752	0.3344128	0.3344128	0.326155264	0.326155264
12	0	0.5	0.4	0.32	0.288	0.2496	0.23936	0.21888	0.2156032	0.2041344	0.203216896	0.1966108672
13	0	0.5	0.4	0.32	0.256	0.2304	0.19968	0.191488	0.1759232	0.17362944	0.165371904	0.1650049024
14	0	0.5	0.4	0.32	0.288	0.2496	0.23936	0.220928	0.2184704	0.20929536	0.20929536	0.2045243392
15	0	0.5	0.4	0.4	0.368	0.368	0.35264	0.354688	0.3473152	0.34960896	0.345938944	0.3478263808
16	1	0.5	0.6	0.6	0.632	0.632	0.64736	0.64736	0.655552	0.655552	0.660270592	0.660270592
17	1	0.5	0.6	0.68	0.712	0.7504	0.76576	0.784192	0.7915648	0.80106752	0.804737536	0.809980416
18	1	0.5	0.6	0.68	0.744	0.7824	0.81312	0.831552	0.8471168	0.85629184	0.864680448	0.8694514688
19	1	0.5	0.6	0.68	0.744	0.7824	0.81312	0.8336	0.8491648	0.8606336	0.868891136	0.8754971648
20	1	0.5	0.6	0.68	0.712	0.7504	0.77088	0.79136	0.8044672	0.815936	0.824062464	0.830616064
21	1	0.5	0.6	0.6	0.632	0.6448	0.66528	0.677568	0.6906752	0.69952256	0.707649024	0.7133987157
22	0	0.5	0.4	0.4	0.4	0.4128	0.42304	0.435328	0.4443392	0.45318656	0.4594343253	0.4652102315
23	0	0.5	0.4	0.4	0.4	0.4128	0.42304	0.43328	0.4422912	0.4490632533	0.4553765547	0.4595941604
24	1	0.5	0.6	0.6	0.632	0.6448	0.66016	0.6704	0.6783189333	0.6852548267	0.6895510756	0.6940395634
25	1	0.5	0.6	0.68	0.712	0.7376	0.75296	0.7625173333	0.7708458667	0.7748144356	0.7797223538	0.7815728962
26	1	0.5	0.6	0.68	0.712	0.7376	0.7461333333	0.7567146667	0.7587171556	0.7640510578	0.764381165	0.7674608716
27	1	0.5	0.6	0.6	0.632	0.6277333333	0.6388266667	0.6342755556	0.6392817778	0.636138477	0.6389298252	0.637070221
28	0	0.5	0.4	0.4	0.3573333333	0.3594666667	0.3395555556	0.3414897778	0.331629037	0.3332735052	0.3282943874	0.3295785693
29	0	0.5	0.4	0.2933333333	0.2666666667	0.2211555556	0.2148977778	0.194797037	0.1939019852	0.1845974914	0.1850165981	0.1805437526
30	0	0.5	0.3333333333	0.2666666667	0.1955555556	0.1777777778	0.147437037	0.1432651852	0.1298646914	0.1292679901	0.1230649942	0.1233443987