Tutorials on Probability Bounds

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Outline

Main Focus: Chernoff bounds and their applications to proving high-probability bounds.

- Basic bounds: Markov’s inequality, Chebyshev’s inequality
- Chernoff bounds
- Examples
- Tree contraction, line breaking (revisited)

Basic Bounds

Markov's inequality

For any random variable $X \geq 0$,

$$\mathbf{Pr}[X > \lambda] < \frac{\mathbf{E}[X]}{\lambda}$$

- If $f : X \to \mathbb{R}_+$ is a positive function, we have

$$\mathbf{Pr}[f(X) > f(\lambda)] < \frac{\mathbf{E}[f(X)]}{f(\lambda)}.$$

- If f is a non-decreasing function, we get that $\mathbf{Pr}[X > \lambda] = \mathbf{Pr}[f(X) > f(\lambda)]$.

(Blackboard interlude)

Basic Bounds (contd.)

Chebyshev's inequality

For any random variable $X \in \mathbb{R}$,

$$\mathbf{Pr}[|X - \mathbf{E}[X]| \geq \lambda] \leq \frac{\mathbf{Var}(X)}{\lambda^2}$$

Chernoff Bounds

- **Let** $X = \sum_i X_i$, where $\mathbf{E}[X_i] = p_i$
  X_i's are *independent*.
- **Will show:** For $0 < \delta < 1$,

$$\mathbf{Pr}[X < (1-\delta)\mu] \leq e^{-\mu\delta^2/2}.$$

Useful facts

- $1 + x \leq e^x$
- $(1-x)\ln(1-x) > -\delta + \delta^2/2$ for $0 < \delta < 1$.

(Blackboard interlude)

Chernoff Bounds

The following bounds can be shown similarly:

1. For $\delta > 0$,

$$\mathbf{Pr}[X < (1-\delta)\mu] < \left(\frac{e^{-\delta}}{(1-\delta)^{1-\delta}}\right)^{\mu}.$$

2. For $0 < \delta < 1$,

$$\mathbf{Pr}[X < (1-\delta)\mu] \leq e^{-\mu\delta^2/2}.$$

3. For any $\delta > 0$,

$$\mathbf{Pr}[X \geq (1+\delta)\mu] < \left(\frac{e^{\delta}}{(1+\delta)^{1+\delta}}\right)^{\mu}.$$

4. For any $0 < \delta \leq 1$,

$$\mathbf{Pr}[X \geq (1+\delta)\mu] \leq e^{-\mu\delta^2/3}.$$

5. For $R \geq 6\mu$,

$$\mathbf{Pr}[X \geq R] \leq 2^{-R}.$$

Randomized (Lazy) Quick Sort

Rand-QSort(A) =

0. If $|A| \leq 1$ **return** A.
1. Pick $e \in A$ uniformly at random.
2. Split A into $A_<$ and $A_>$.
3. Go to Step 1 if $\max\{|A_<|, |A_>|\} > \frac{3}{4}|A|$.
4. **return**
   Rand-QSort$(A_<)$++$\{$e$\}$++Rand-QSort$(A_>)$

Parallel Tree Contraction (Miller and Reif, 1985)

Each contraction step performs **rake** and **compress** (in the order you wish):

- **Rake**. Contract all leaves into parent nodes.
- **Compress**. Shorten long chains of one-child nodes by employing a standard coin-flipping algorithm. All nodes flip coins; if a node comes up “heads”, and has a single child that comes up “tails”, it contracts that child into itself.

In Your Homework: 1) Each contraction removes at least a constant fraction of nodes *w.h.p.* 2) The algorithm terminates in $O(\log n)$ steps *w.h.p.*

Line breaking

Two major components:

- If I start a line, who should start the next line?
- You have a tree, what do you do with it?