Lecture 9:

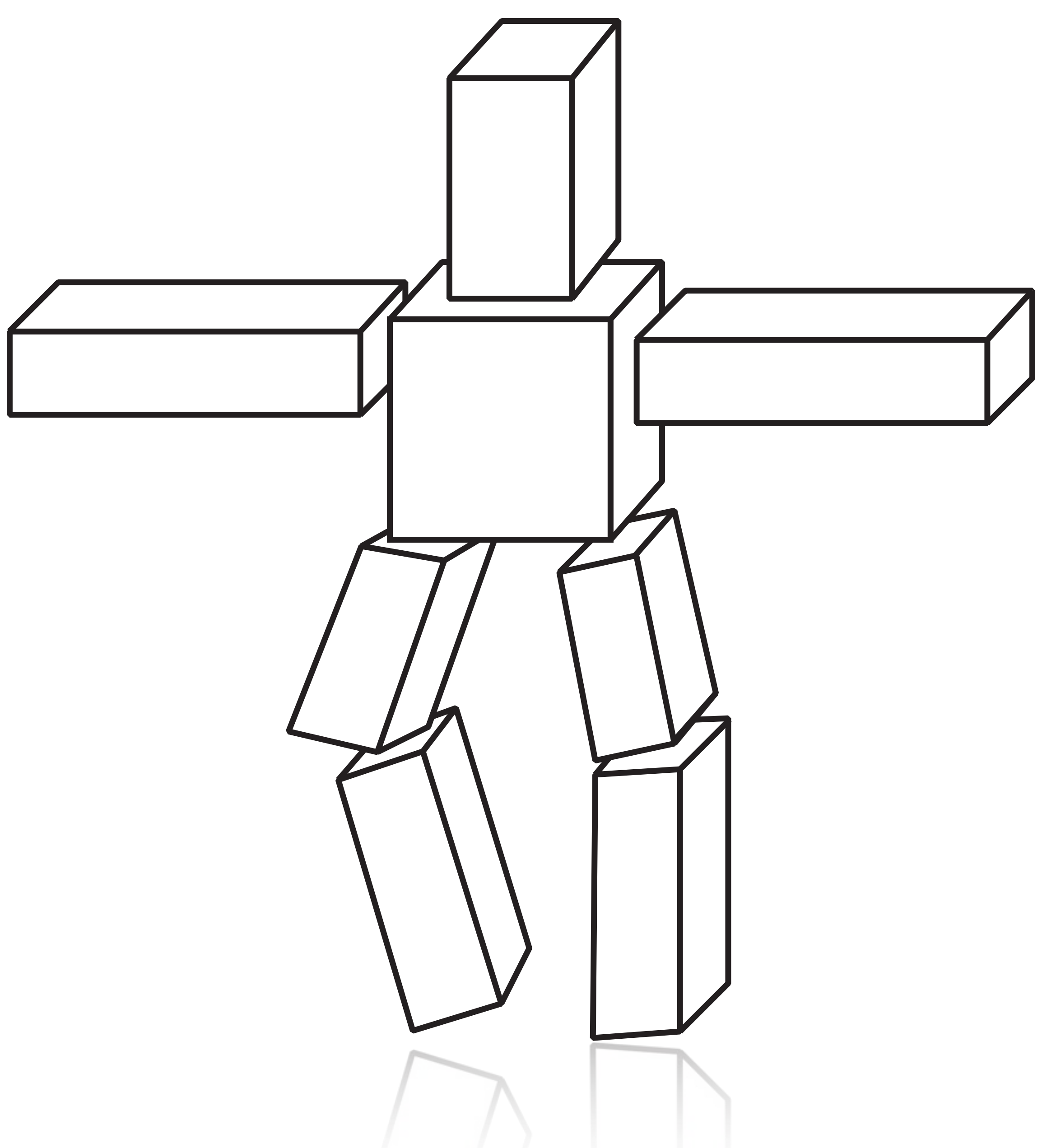
Introduction to Geometry

Computer Graphics

CMU 15-462/15-662, Spring 2016

Increasing the complexity of our models

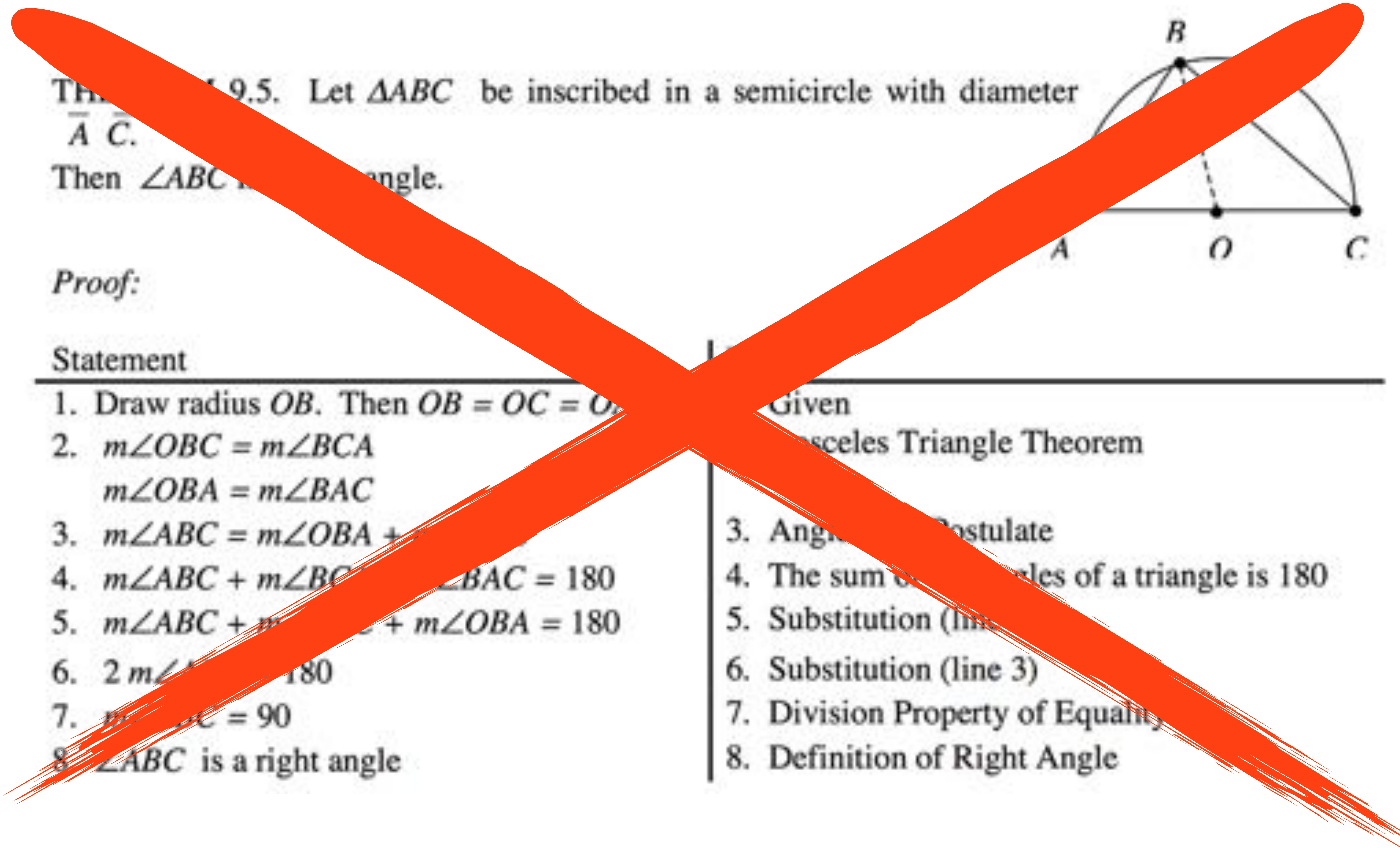
Transformations



Geometry

Materials, lighting, ...

What is geometry?



Ceci n'est pas géométrie.

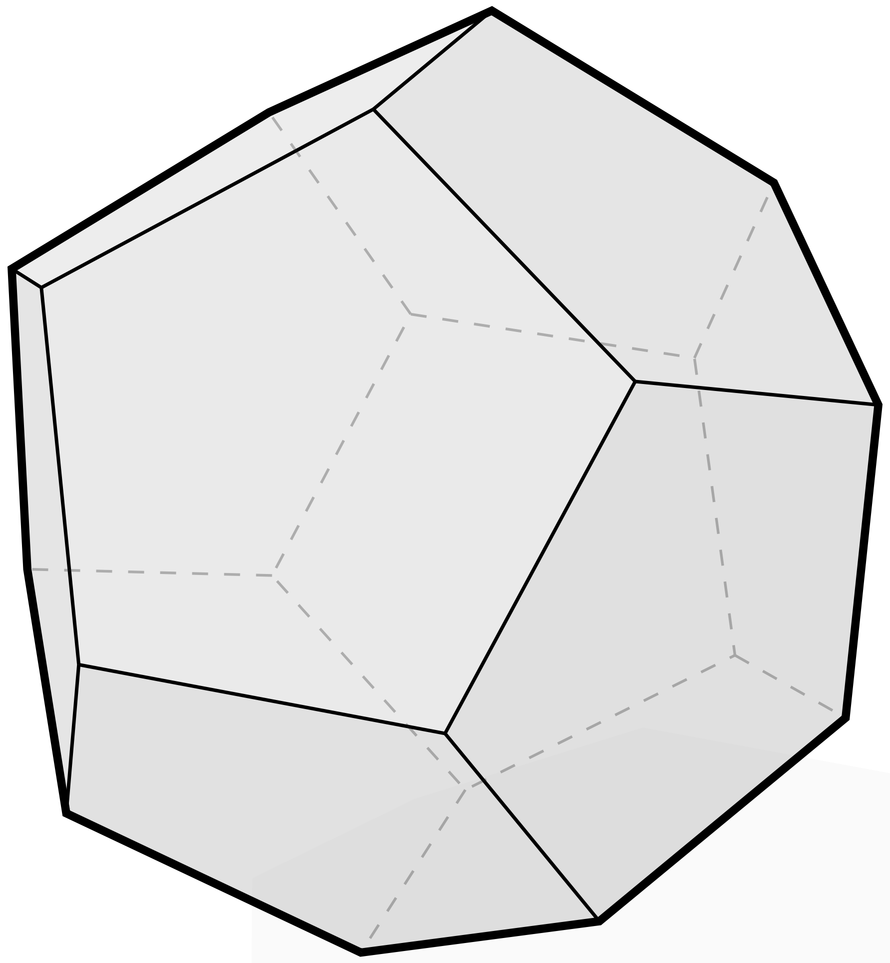
See: Paul Lockhart, "A Mathematician's Lament"

What is geometry?

“Earth” “measure”

ge•om•et•ry /jēˈämətrē/ *n.*

1. The study of shapes, sizes, patterns, and positions.
2. The study of spaces where some quantity (lengths, angles, etc.) can be *measured*.



Plato: “...the earth is in appearance like one of those balls which have leather coverings in twelve pieces...”

How can we describe geometry?

IMPLICIT

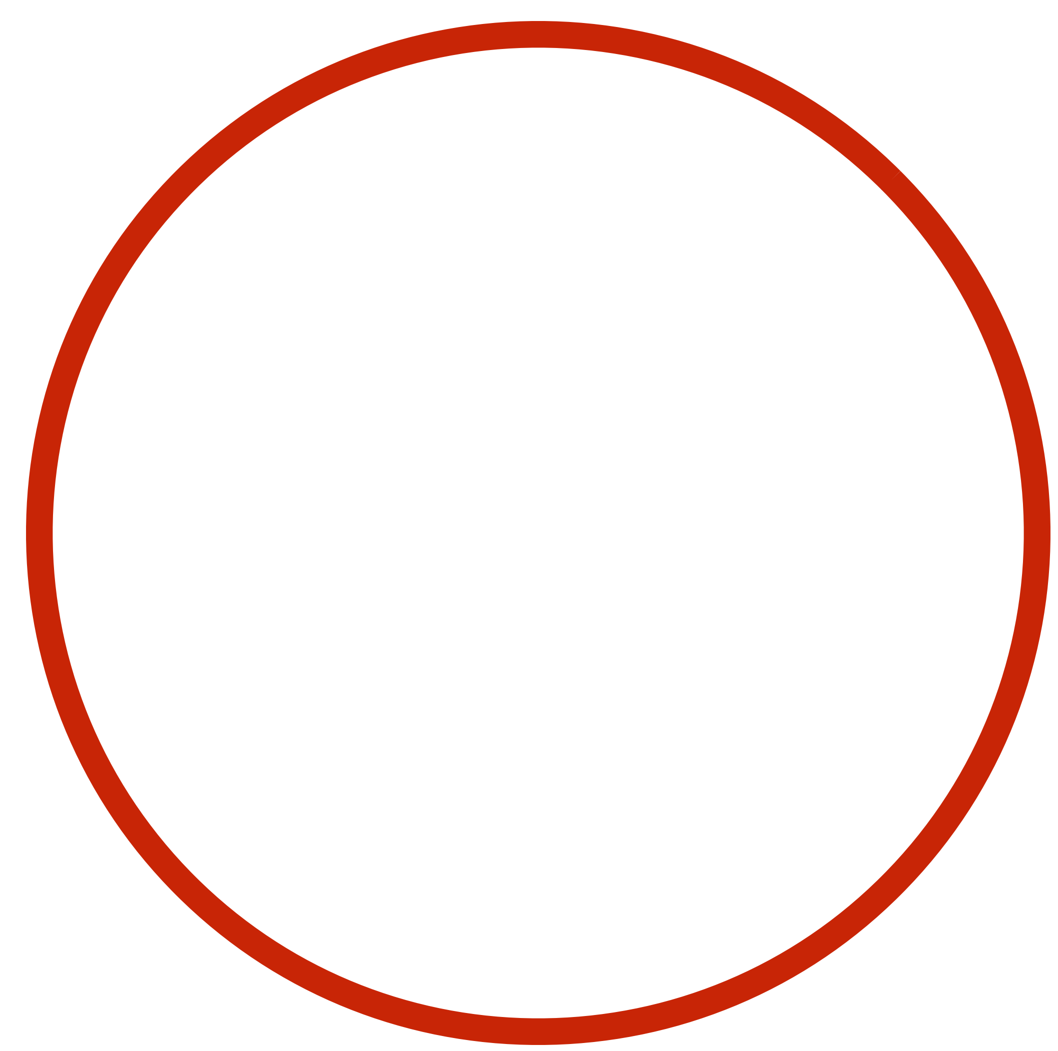
$$x^2+y^2=0$$

LINGUISTIC

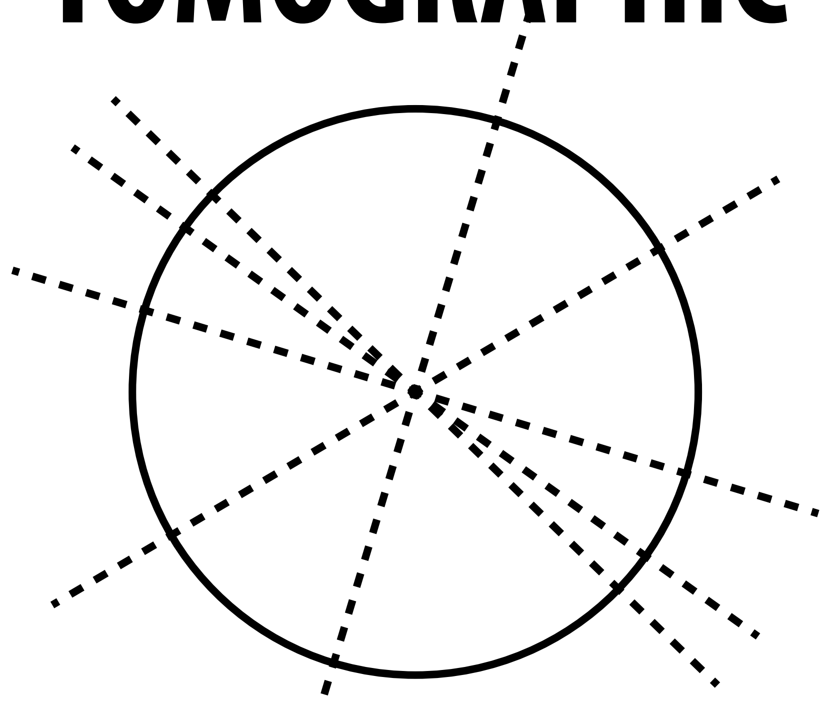
"unit circle"

EXPLICIT

$$(\underbrace{\cos\theta}_{x}, \underbrace{\sin\theta}_{y})$$



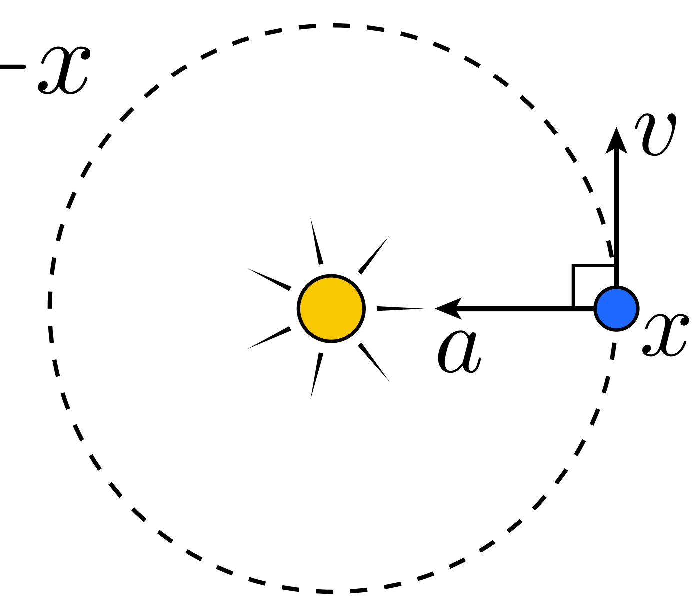
TOMOGRAPHIC



(constant density)

DYNAMIC

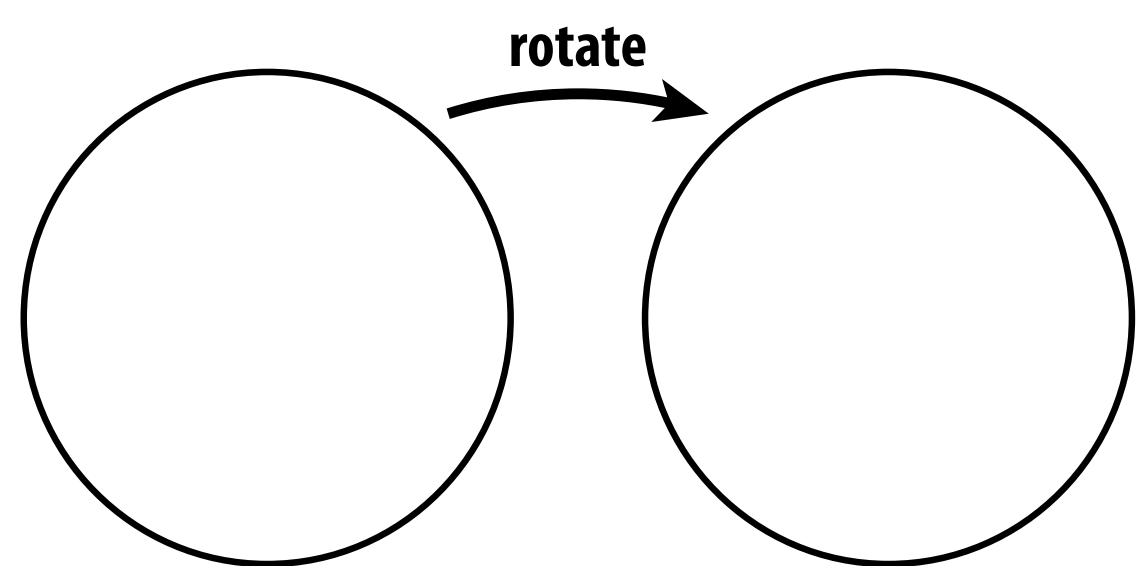
$$\frac{d^2}{dt^2}x = -x$$



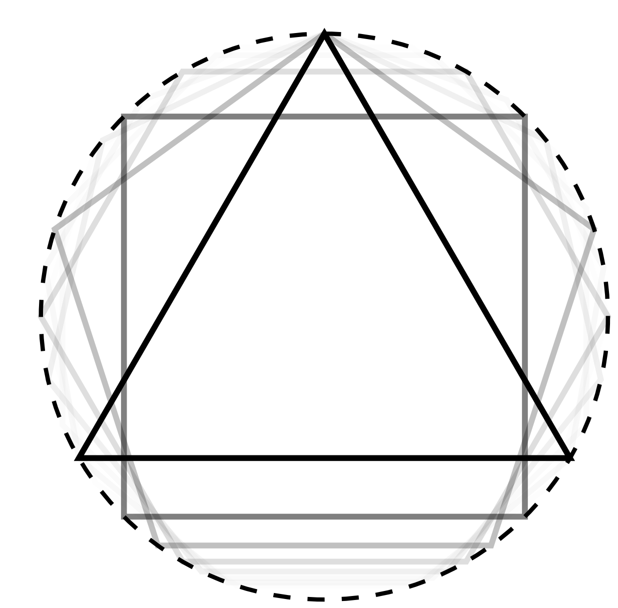
CURVATURE

$$\kappa = 1$$

SYMMETRIC



DISCRETE

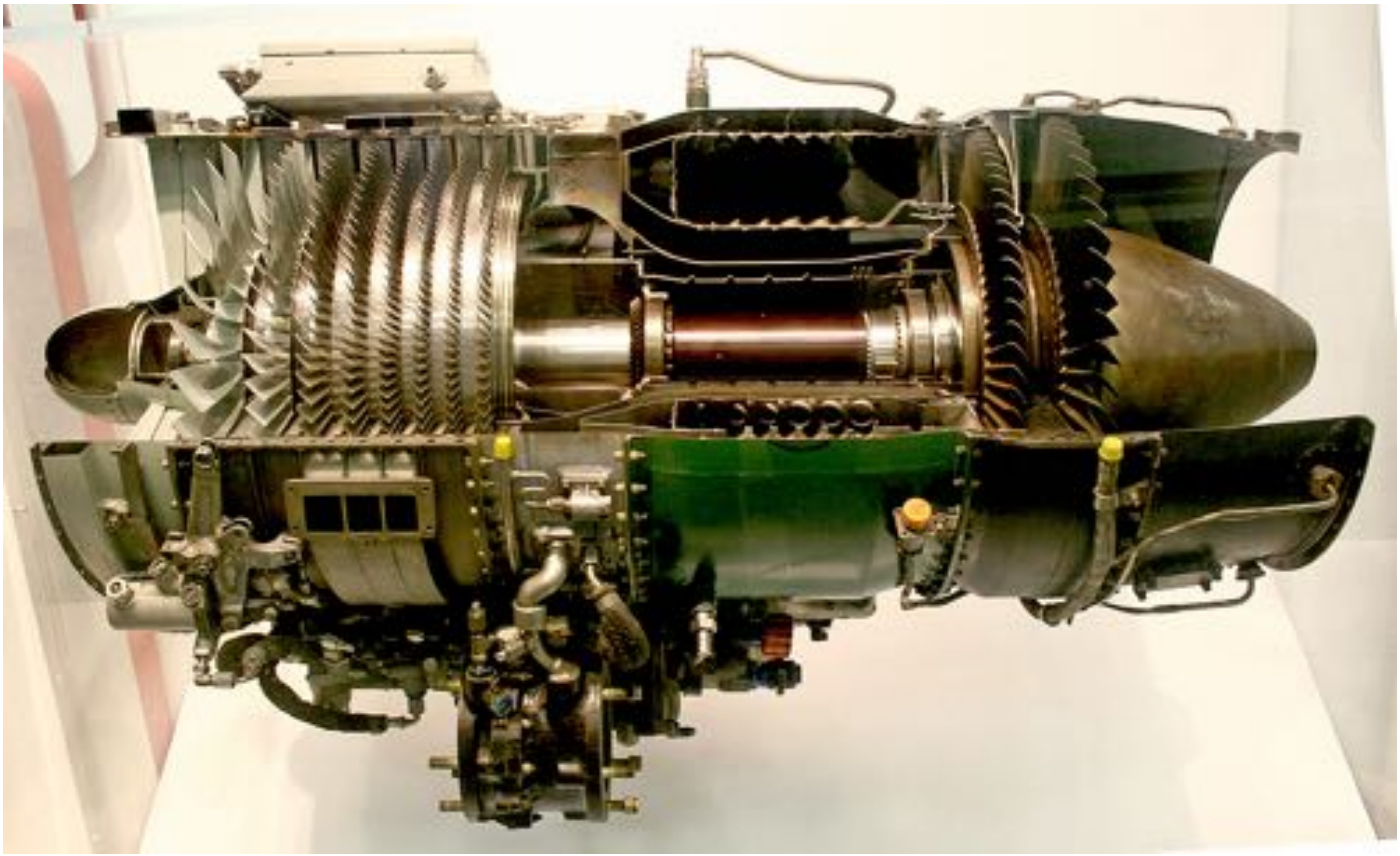


$n \to \infty$

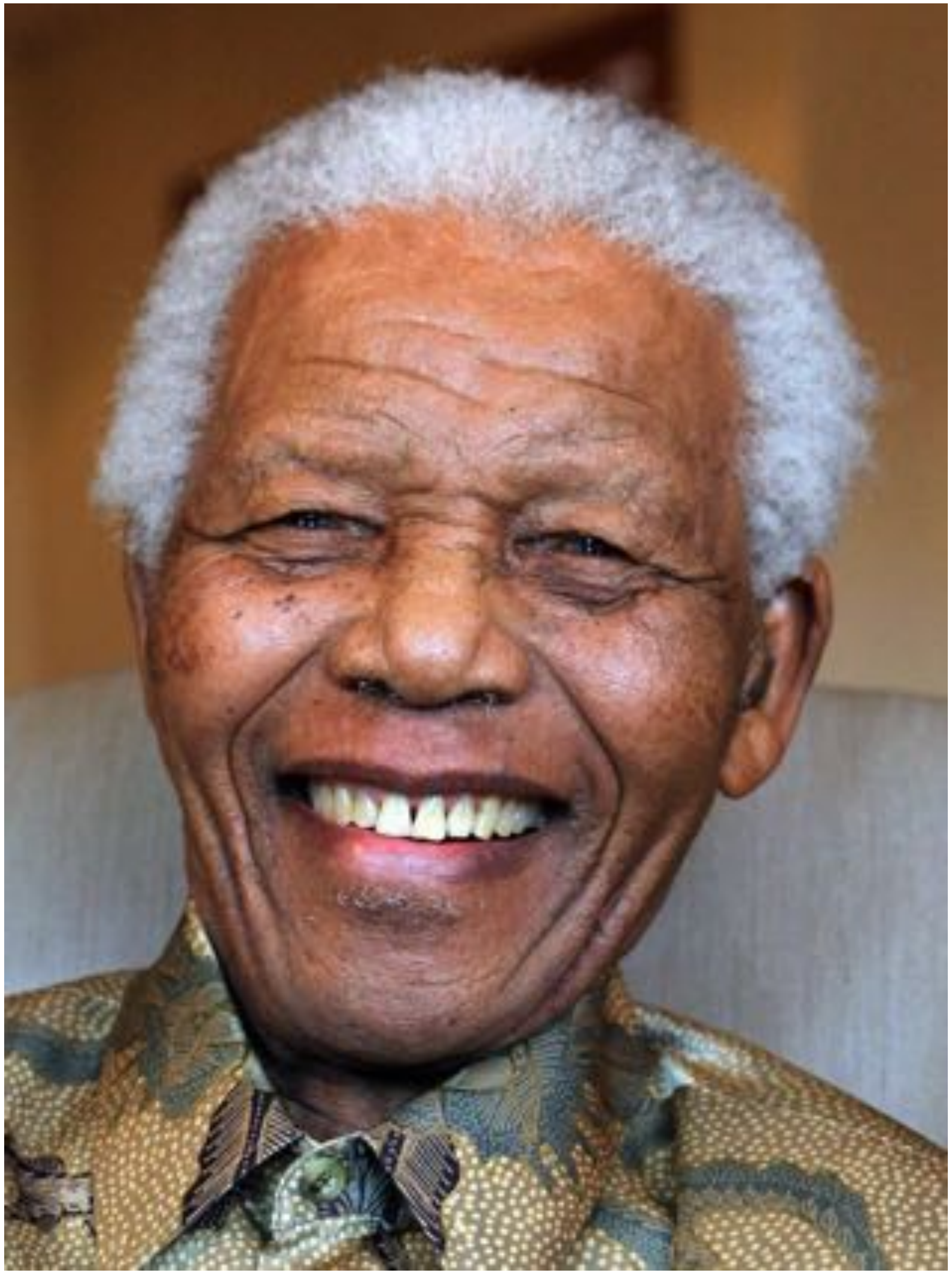
Given all these options, what's the best way to encode geometry on a computer?

Examples of geometry

Examples of geometry



Examples of geometry



Examples of geometry

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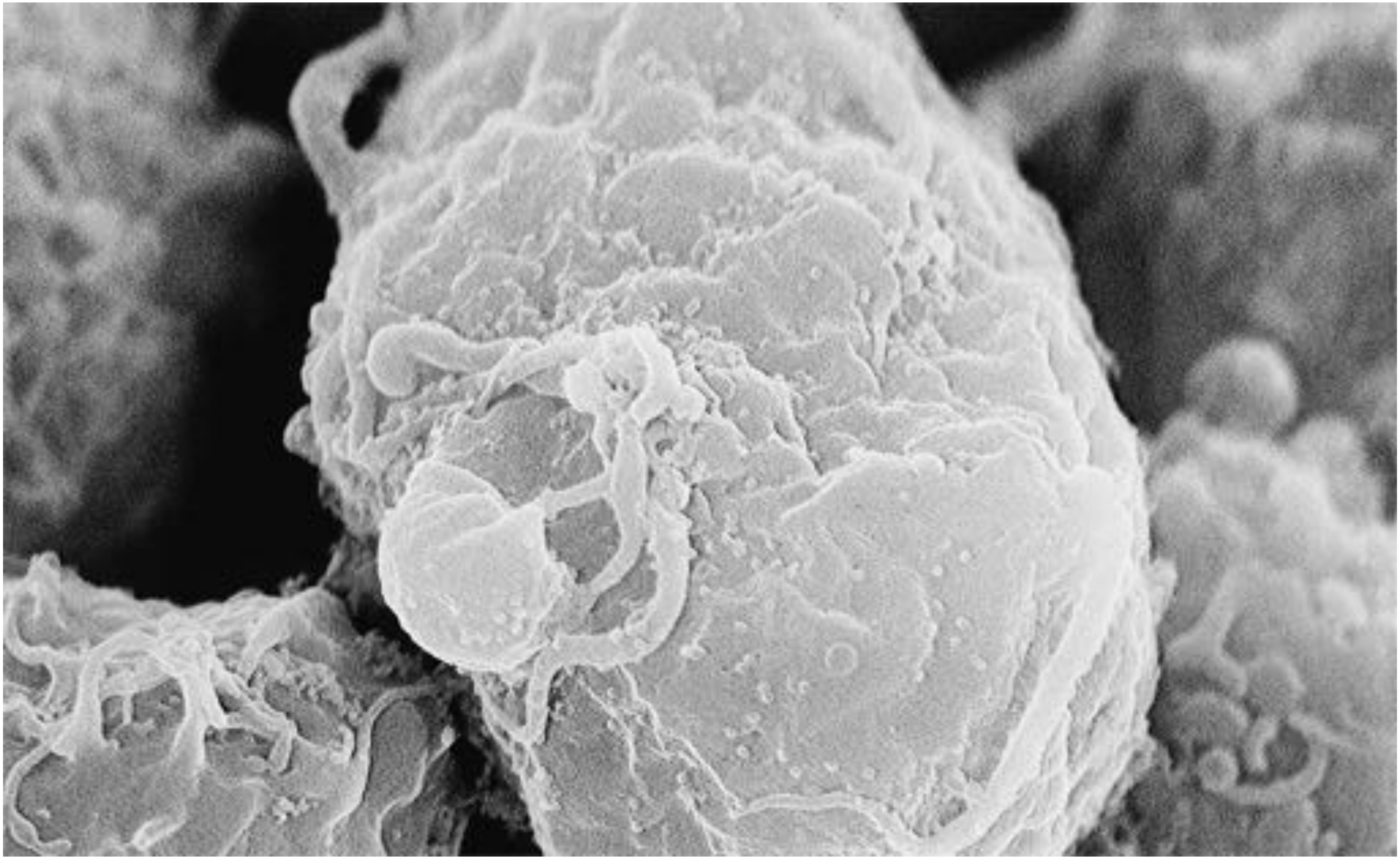
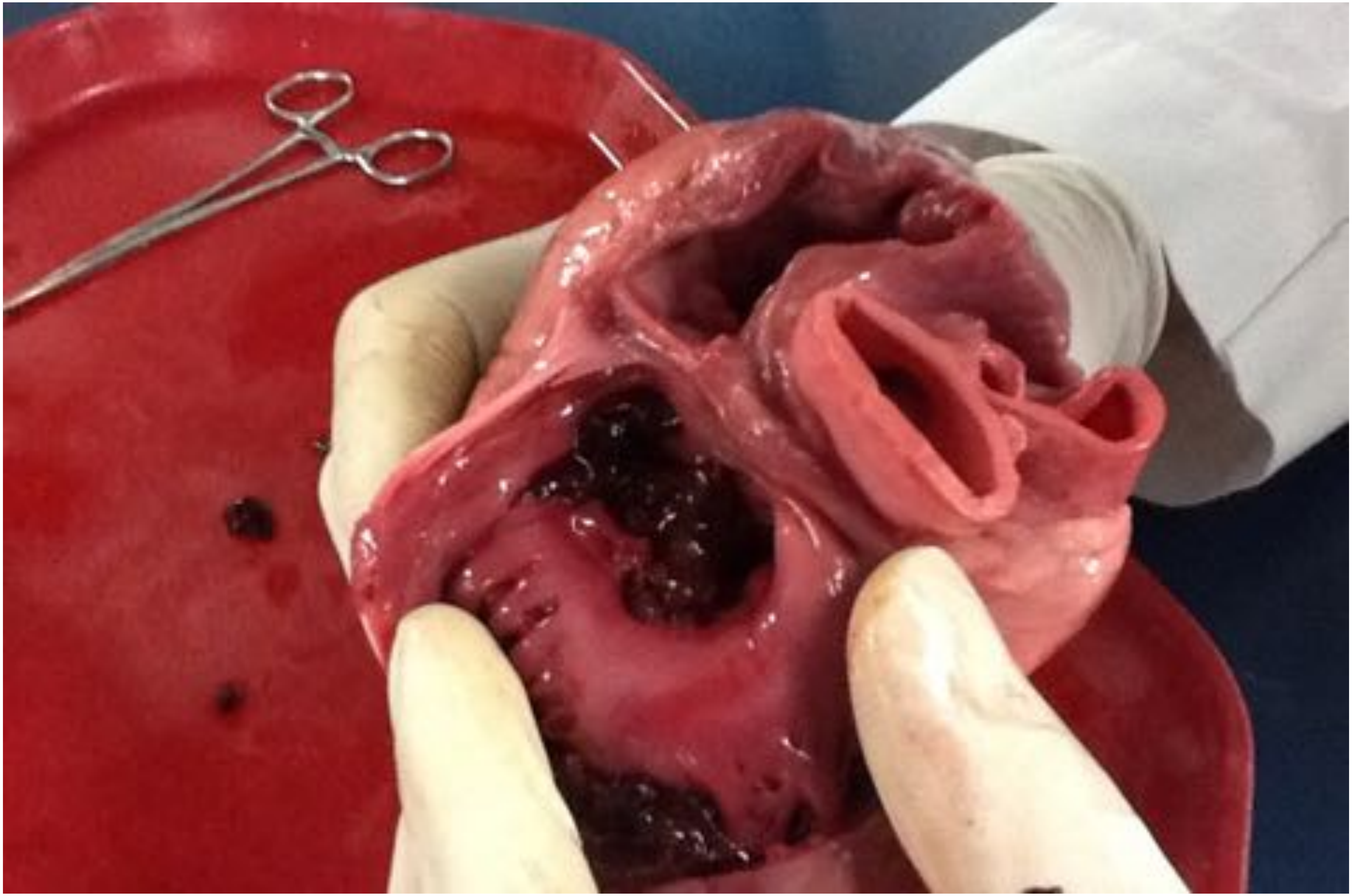


Examples of geometry



Examples of geometry

No one "best" choice—geometry is hard!

"I hate meshes.
I cannot believe how hard this is.
Geometry is hard."

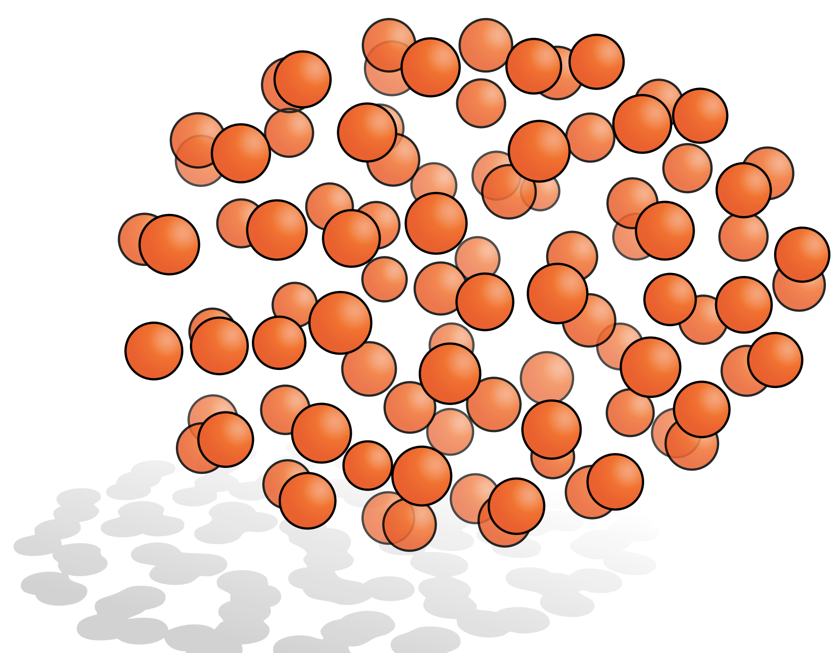
—David Baraff

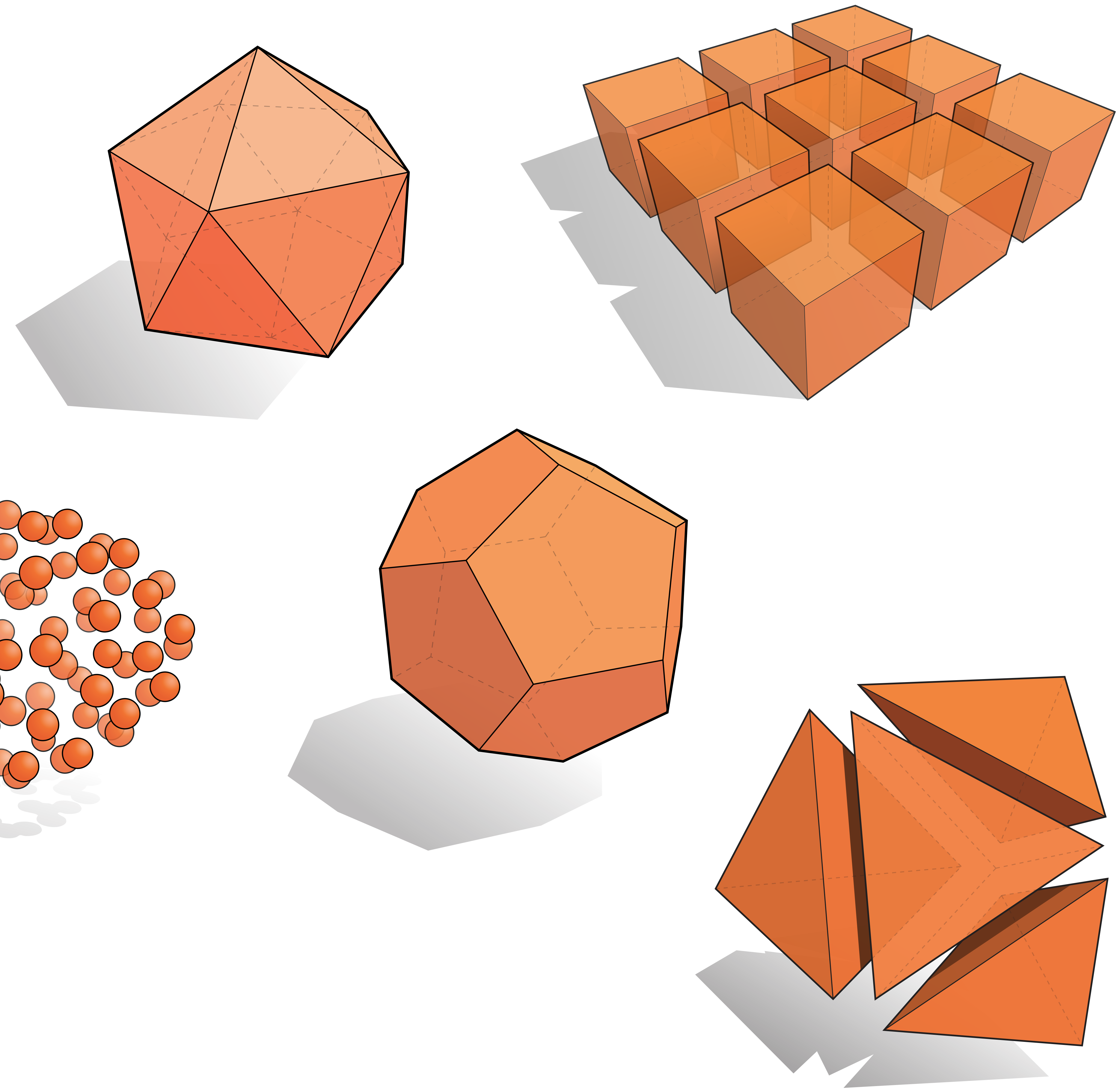
Senior Research Scientist
Pixar Animation Studios

Slide cribbed from Jeff Erickson.

Many ways to digitally encode geometry

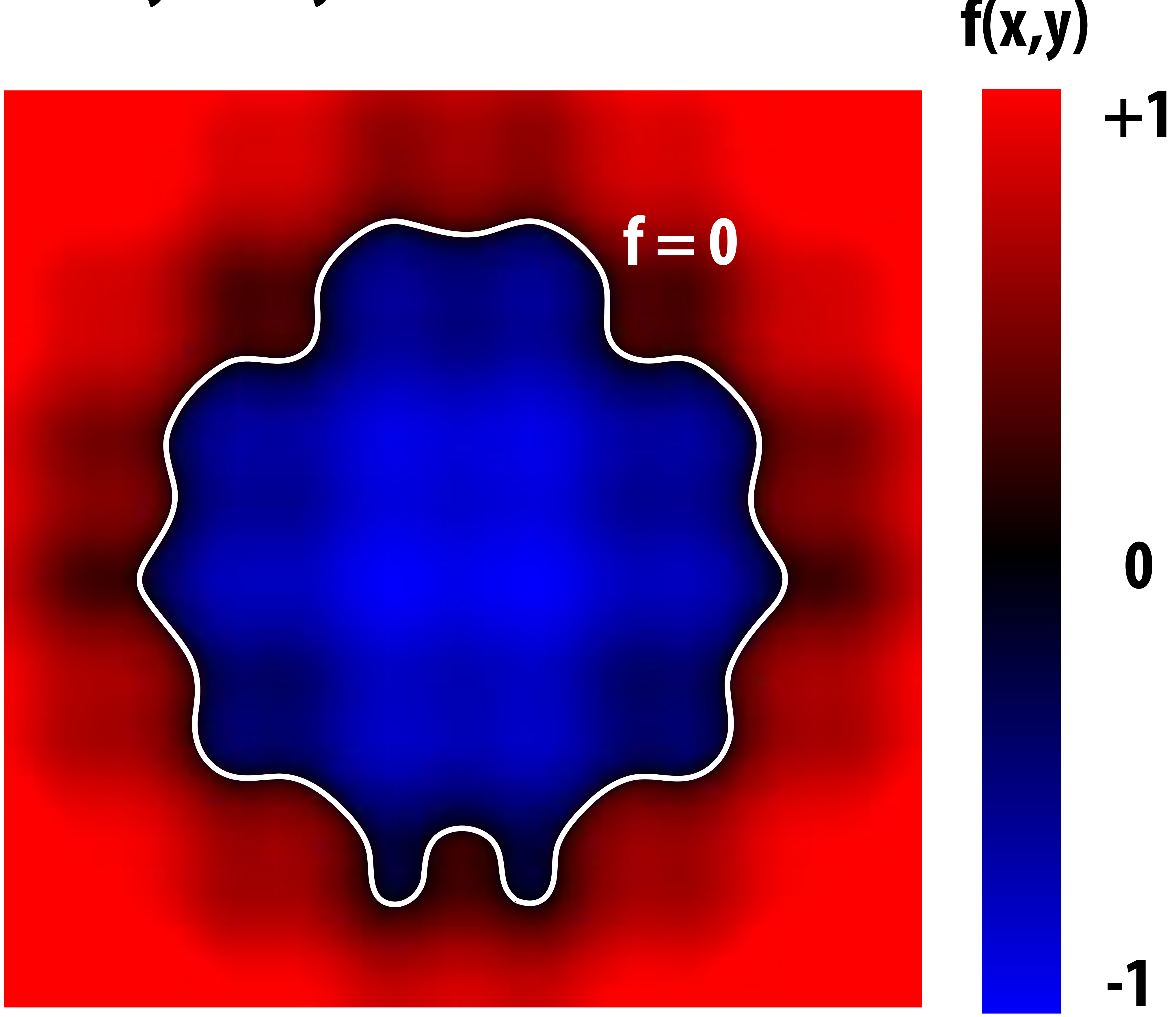
- **EXPLICIT**
  - point cloud
  - polygon mesh
  - subdivision, NURBS
  - L-systems
  - ...
- **IMPLICIT**
  - level set
  - algebraic surface
  - ...
- **Each choice best suited to a different task/type of**





"Implicit" Representations of Geometry

- Points aren't known directly, but satisfy some relationship
- E.g., unit sphere is all points x such that $x^2+y^2+z^2=1$
- More generally, $f(x,y,z) = 0$



Many implicit representations in graphics

- algebraic surfaces
- constructive solid geometry
- level set methods
- blobby surfaces
- fractals





(Will see some of these a bit later.)

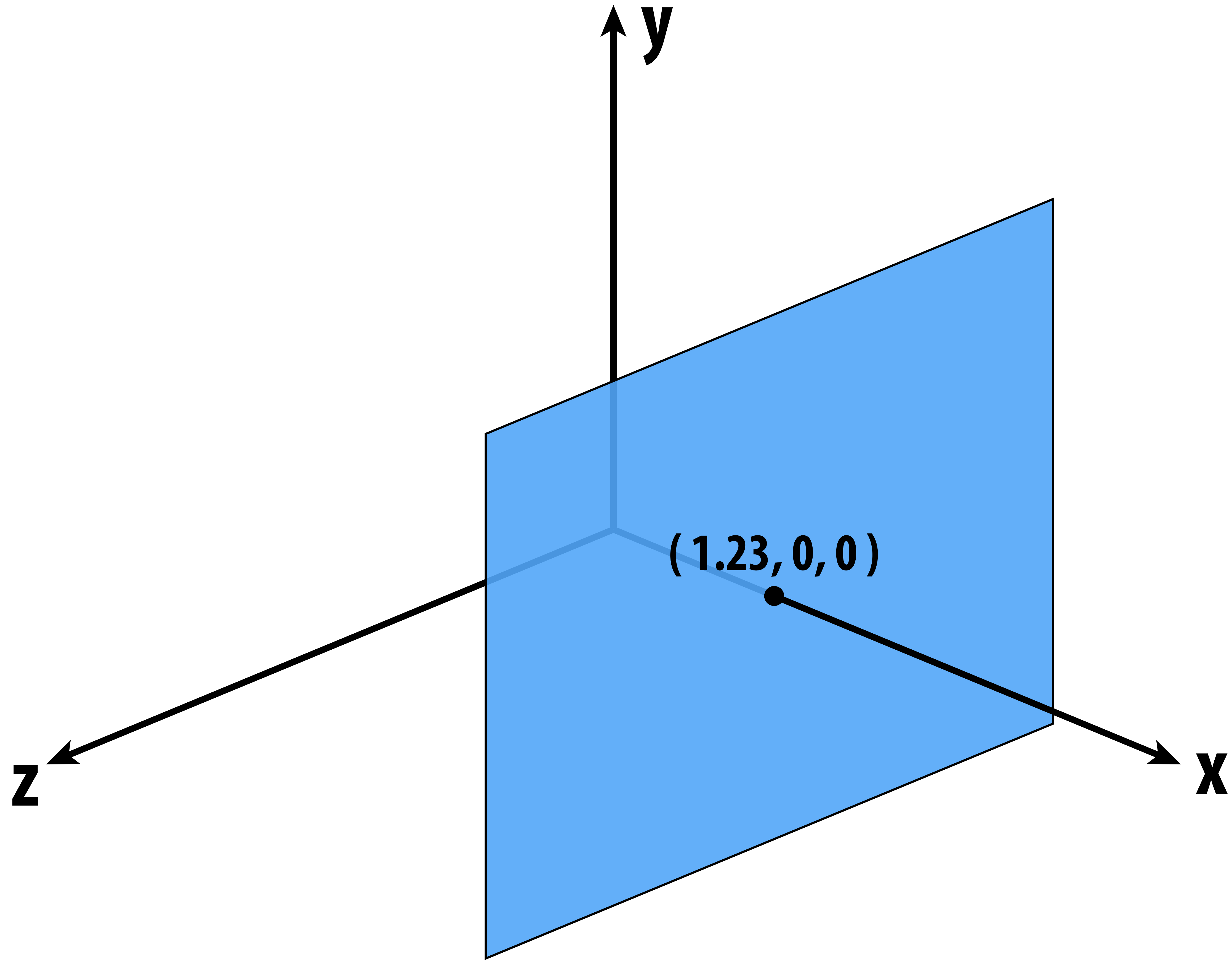
But first, let's play a game:

I'm thinking of an implicit surface $f(x,y,z)=0$.

Find any point on it.

Give up?

My function was (x -1.23 (a plane):



Implicit surfaces make some tasks hard (like sampling).

Let's play another game.

I have a new surface $f(x,y,z) = x^2 + y^2 + z^2 - 1$

I want to see if a point is inside it.

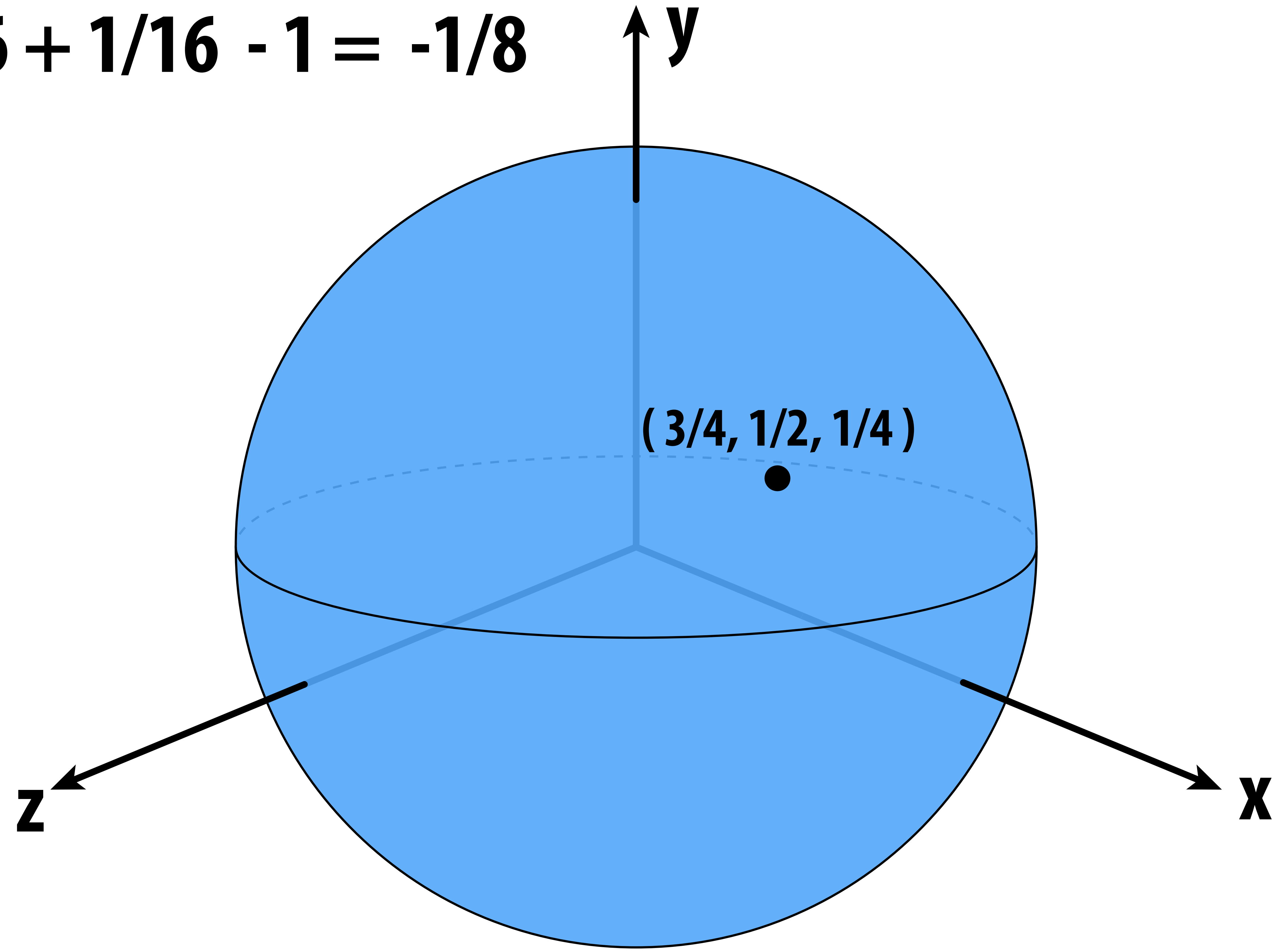
Check if this point is inside the unit sphere

How about the point (3/4, 1/2, 1/4)?

9/16 + 4/16 + 1/16 - 1 = -1/8

-1/8 < 0

YES.



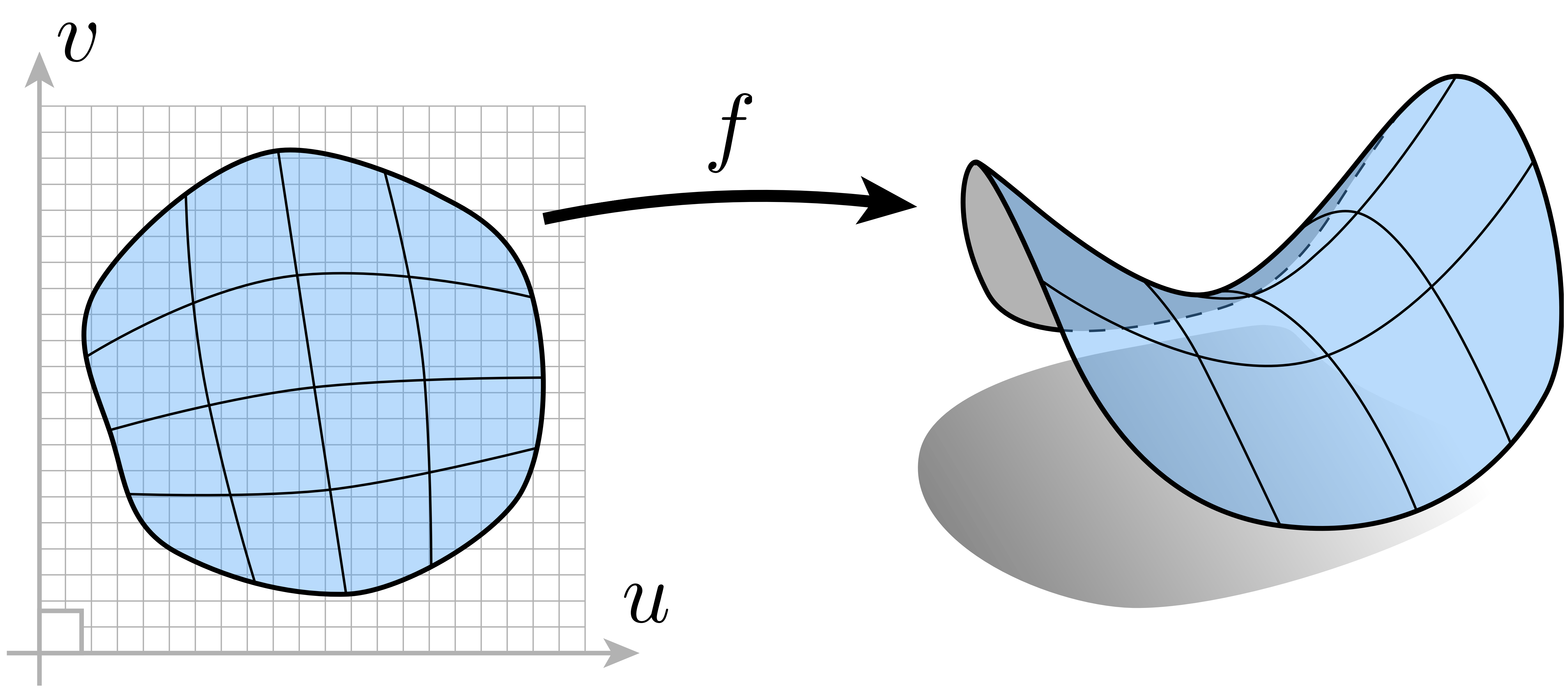
Implicit surfaces make other tasks easy (like inside/outside tests).

"Explicit" Representations of Geometry

- **All points are given directly**
- **E.g., points on sphere are**

$$(\cos(u)\sin(v), \sin(u)\sin(v), \cos(v)),$$
$$\text{for } 0 \leq u < 2\pi \text{ and } 0 \leq v \leq \pi$$

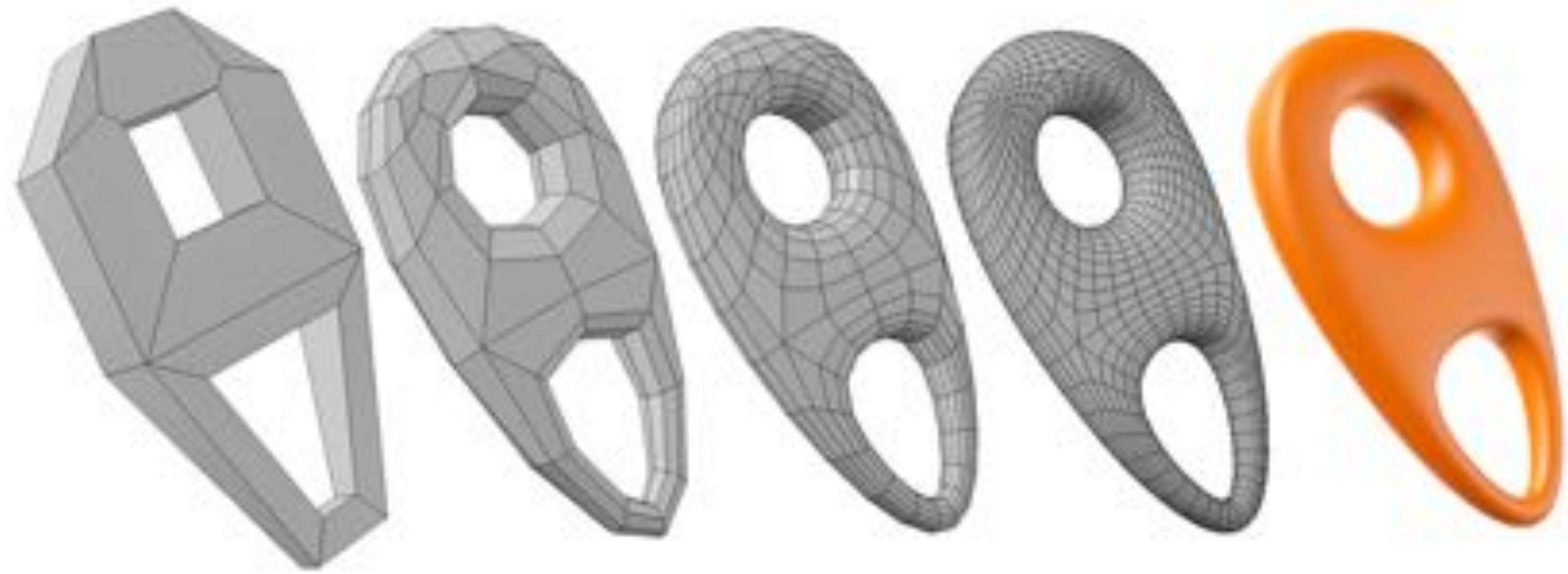
- **More generally:** $f : \mathbb{R}^2 \to \mathbb{R}^3; (u, v) \mapsto (x, y, z)$

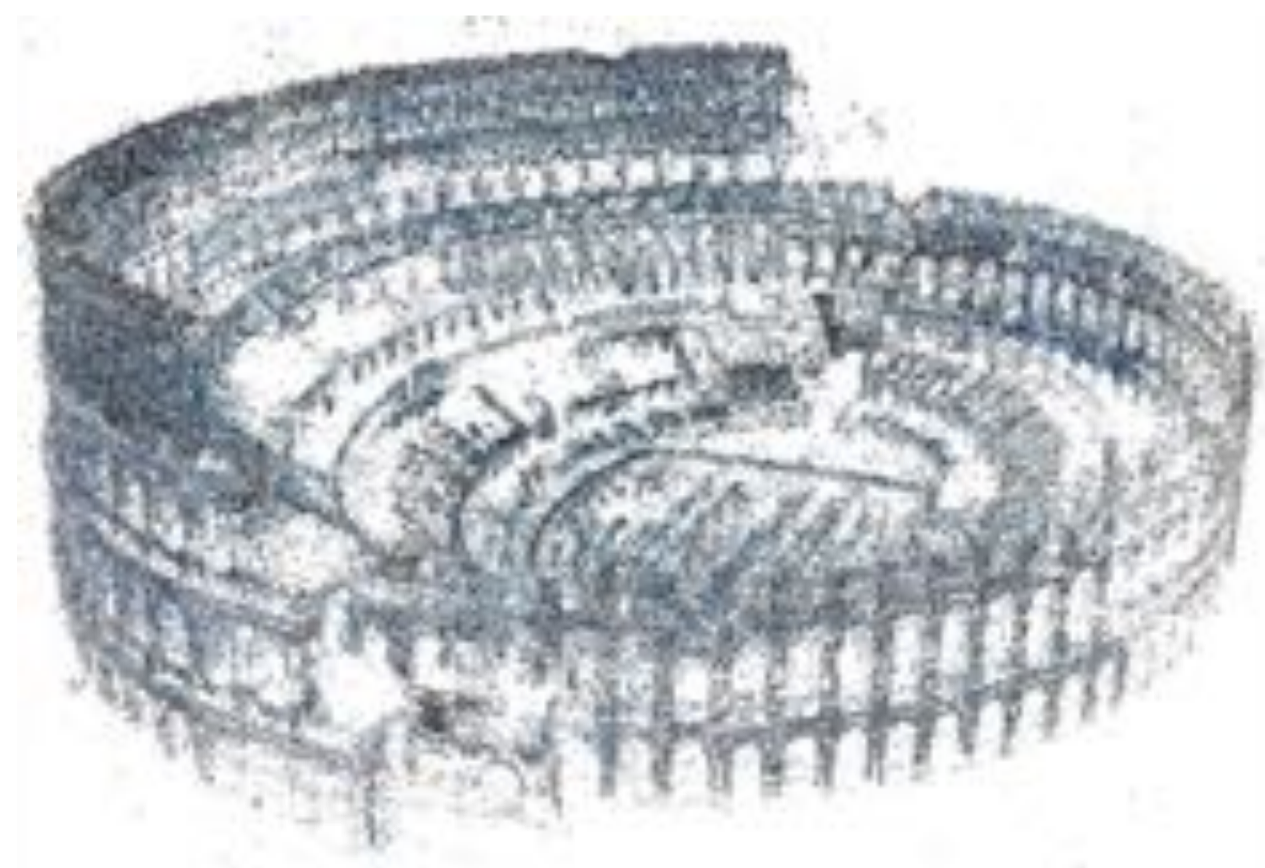


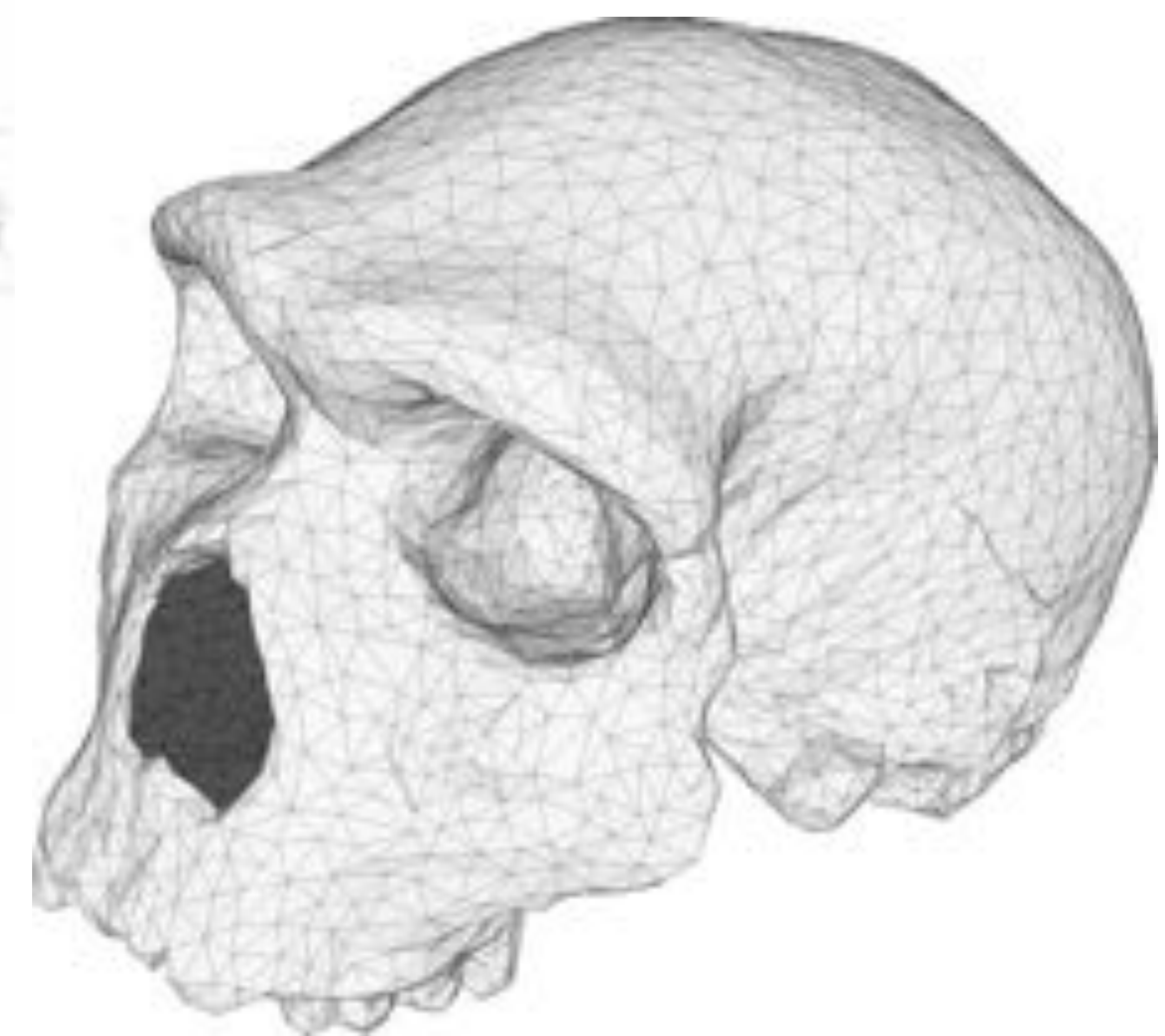
- **(Might have a bunch of these maps, e.g., one per triangle.)**

Many explicit representations in graphics

- triangle meshes
- polygon meshes
- subdivision surfaces
- NURBS
- point clouds
- ...







(Will see some of these a bit later.)

But first, let's play a game:

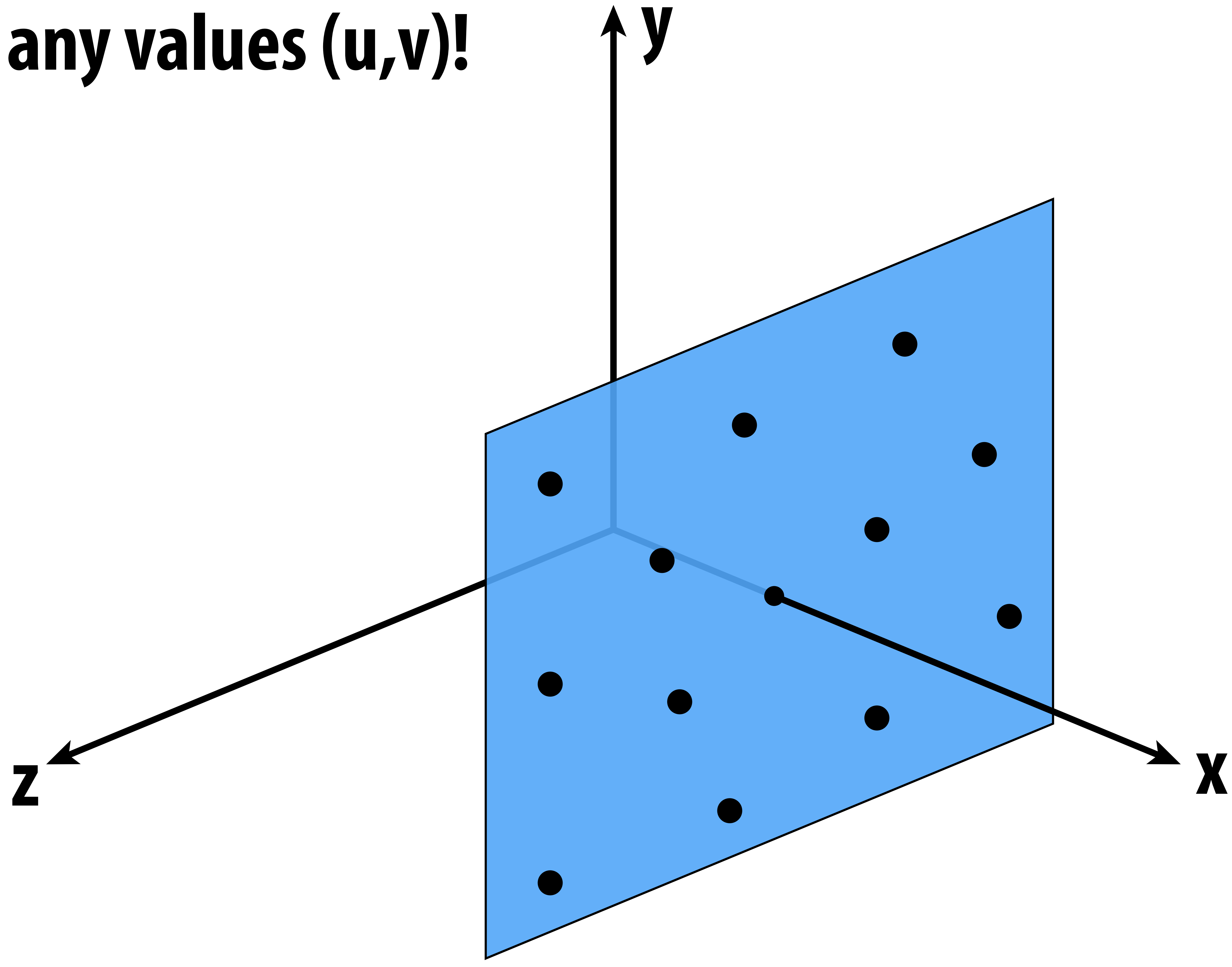
I'll give you an explicit surface.

You give me some points on it.

Sampling an explicit surface

My surface is f(u, v) = (1.23, u, v).

Just plug in any values (u,v)!



Explicit surfaces make some tasks easy (like sampling).

Let's play another game.

I have a new surface f(u,v).

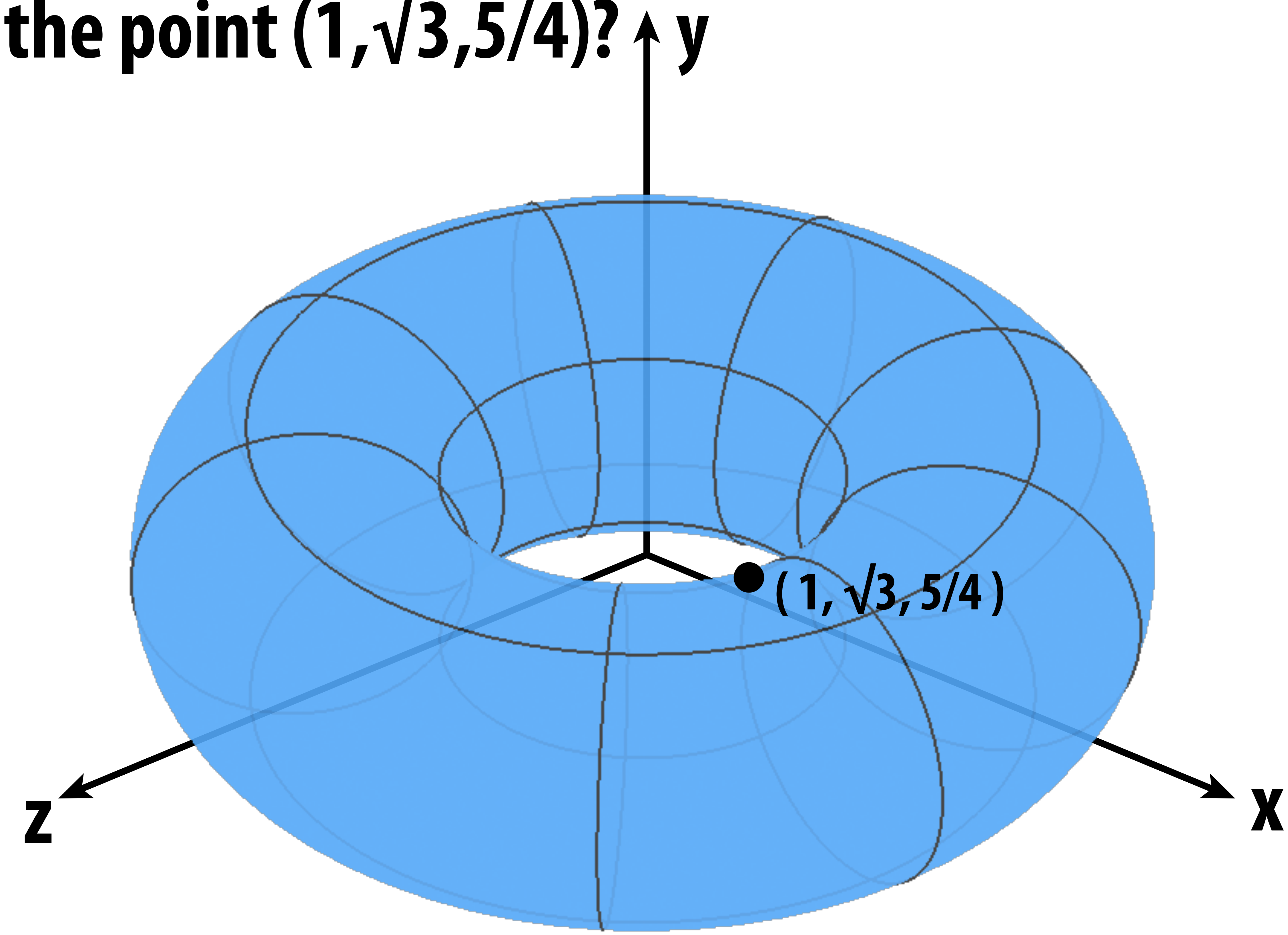
I want to see if a point is inside it.

Check if this point is inside the torus

My surface is $f(u,v) = ((2+\cos(u))\cos(v), (2+\cos(u))\sin(v), \sin(u))$

How about the point $(1,\sqrt{3},5/4)$?

...NO!



Explicit surfaces make other tasks hard (like inside/outside tests).

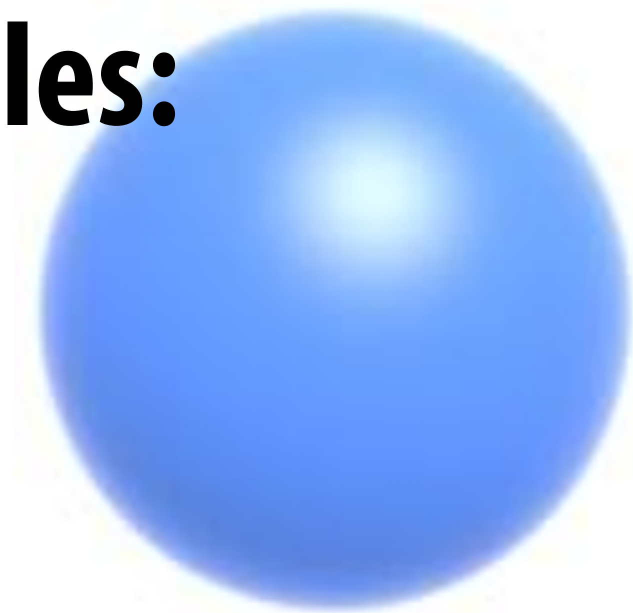
CONCLUSION:
Some representations work better than others—depends on the task!

Different representations will also be better suited to different types of geometry.

Let's take a look at some common representations used in computer graphics.

Algebraic Surfaces (Implicit)

- **Surface is zero set of a polynomial in x, y, z ("algebraic variety")**
- **Examples:**



$x^2+y^2+z^2=1$

$(R-\sqrt{x^2+y^2})^2+z^2=r^2$

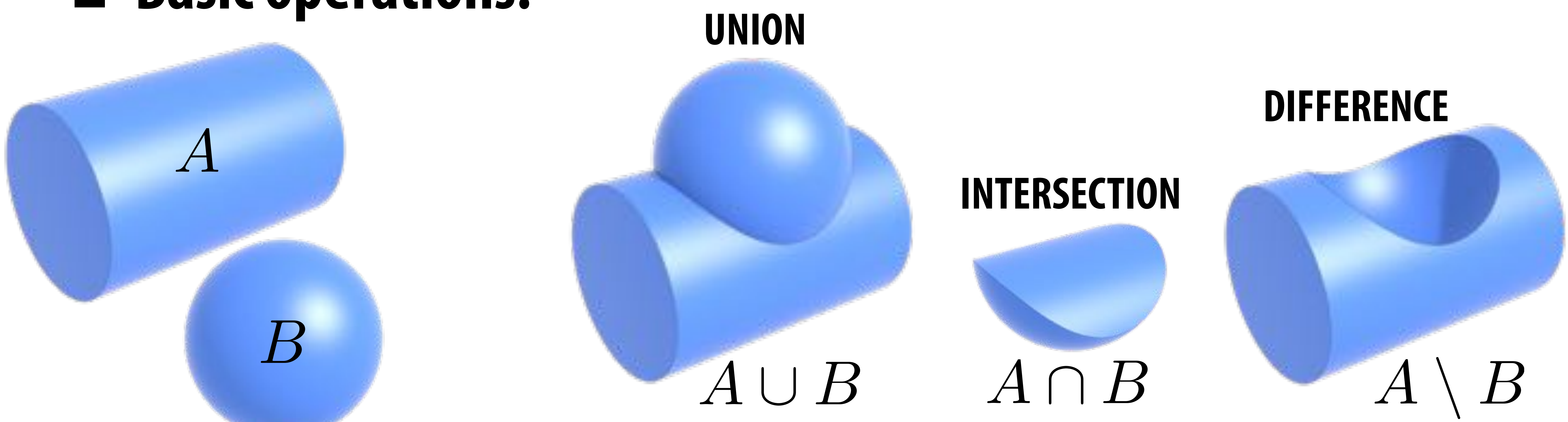
$(x^2+\frac{9y^2}{4}+z^2-1)^3 = x^2z^3+\frac{9y^2z^3}{80}$

- **What about more complicated shapes?**

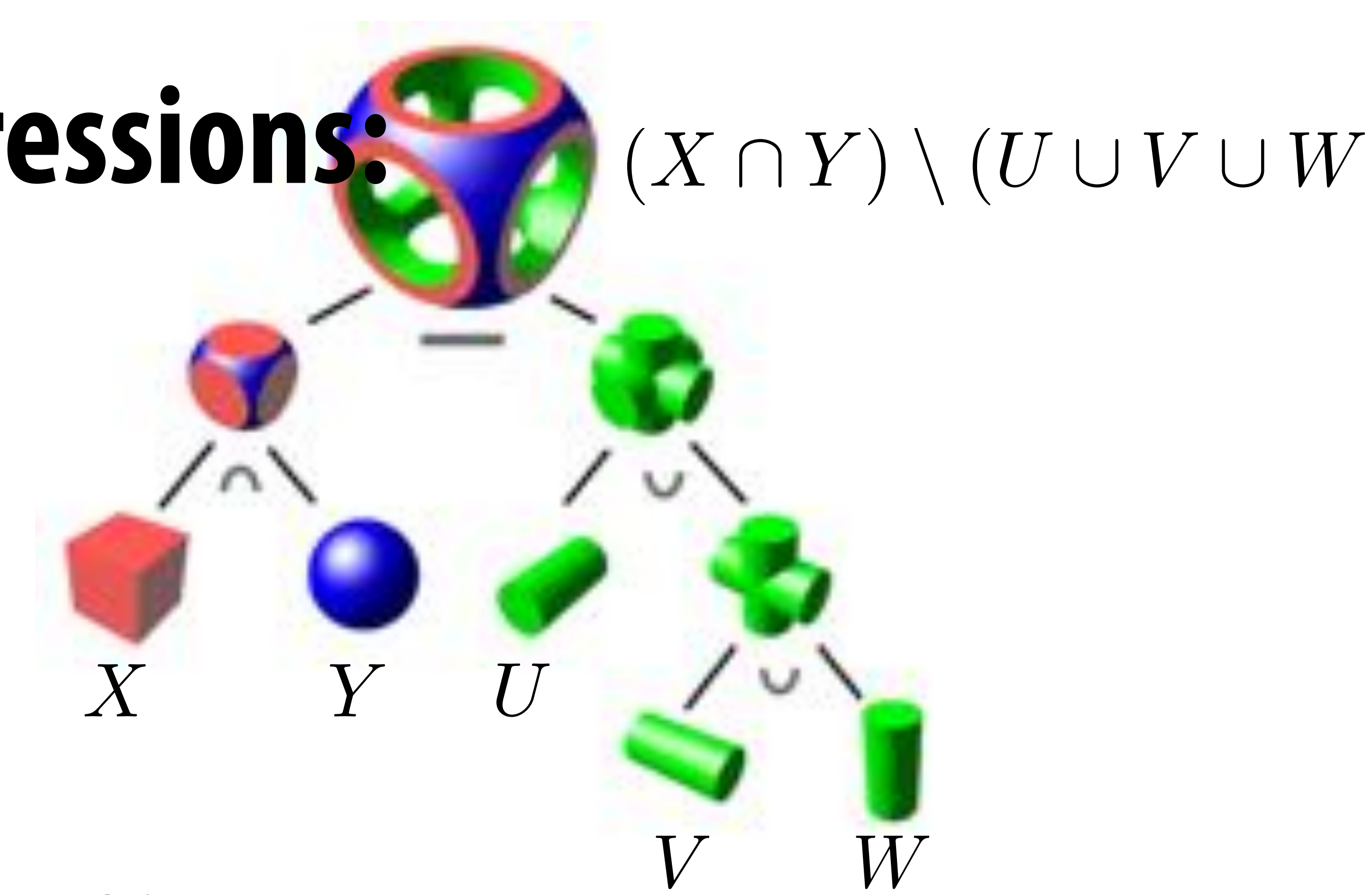
- **Very hard to come up with polynomials!**

Constructive Solid Geometry (Implicit)

- Build more complicated shapes via Boolean operations
- Basic operations:

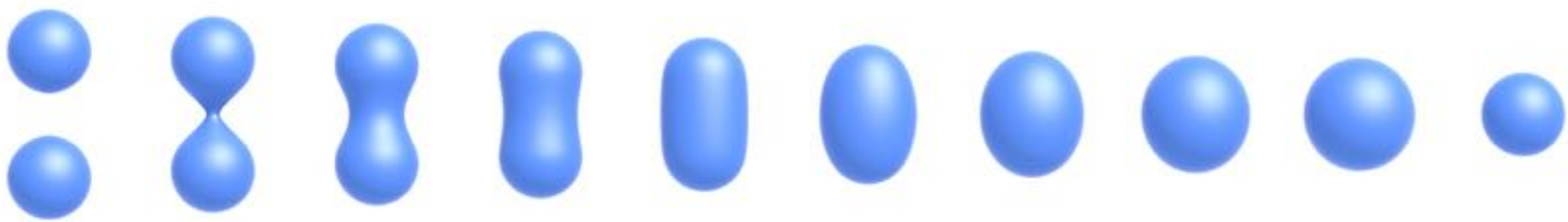


- Then chain together expressions:

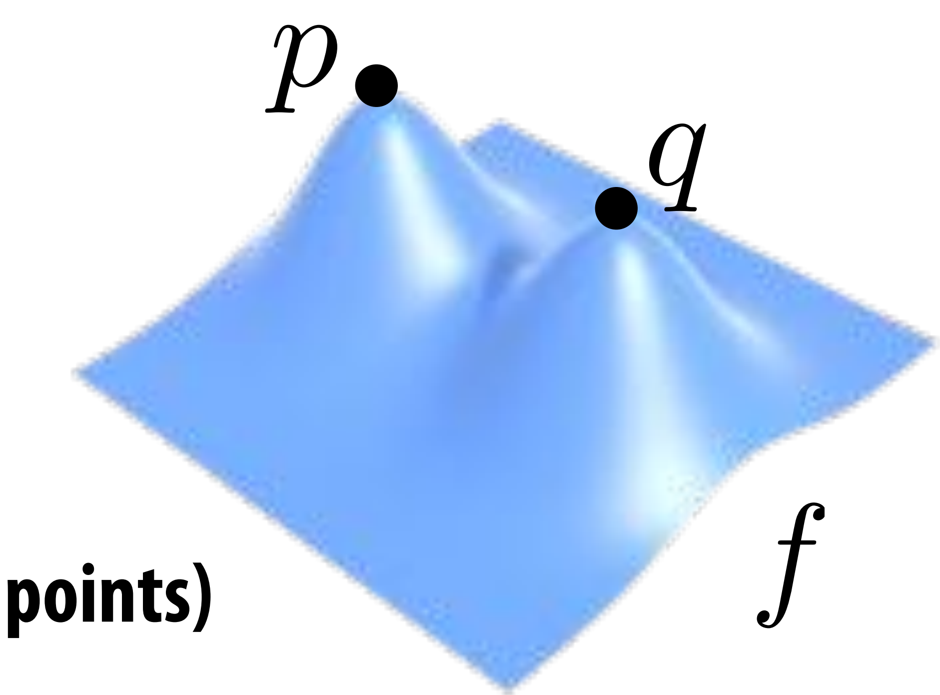


Blobby Surfaces (Implicit)

- **Instead of Booleans, gradually blend surfaces together:**

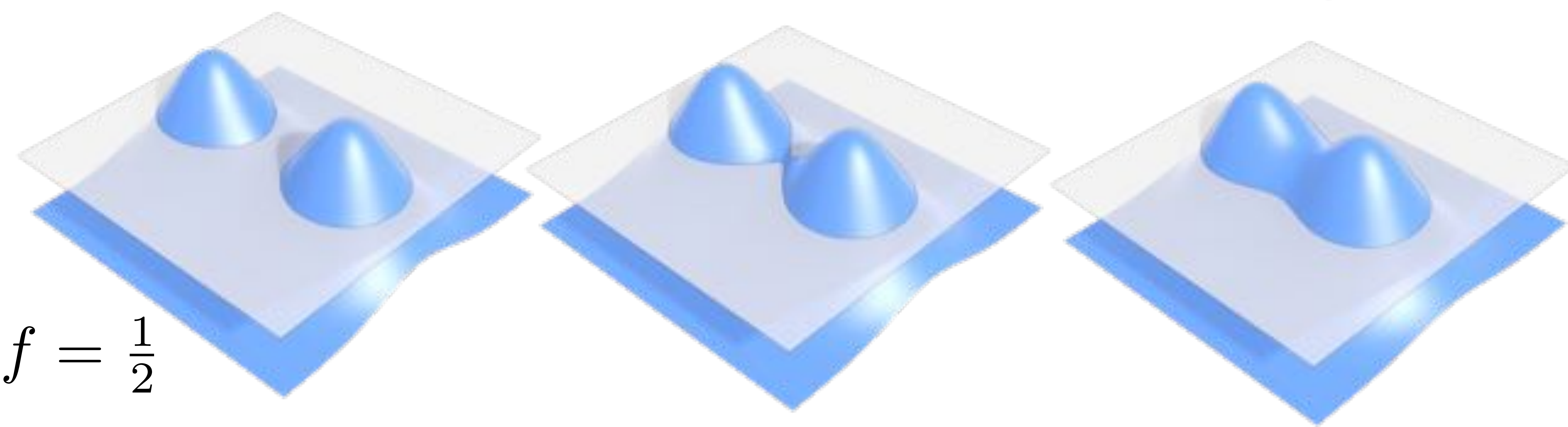


- **Easier to understand in 2D:**



$\phi_p(x) := e^{|x-p|^2}$ **(Gaussian centered at p)**

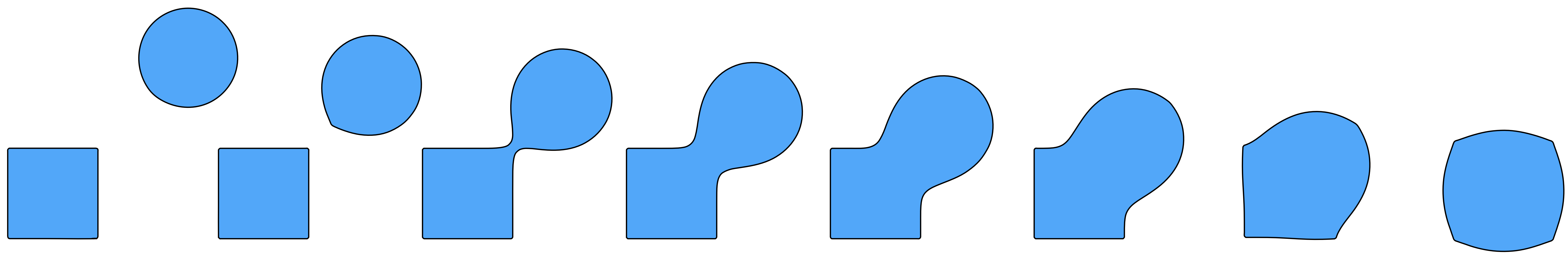
$f := \phi_p + \phi_q$ **(Sum of Gaussians centered at different points)**



$f = \frac{1}{2}$

Blending Distance Functions (Implicit)

- A distance function gives distance to closest point on object
- Can blend any two distance functions d_1, d_2:



- Similar strategy to points, though many possibilities. E.g.,

$$f(x) := e^{d_1(x)^2} + e^{d_2(x)^2} - \frac{1}{2}$$

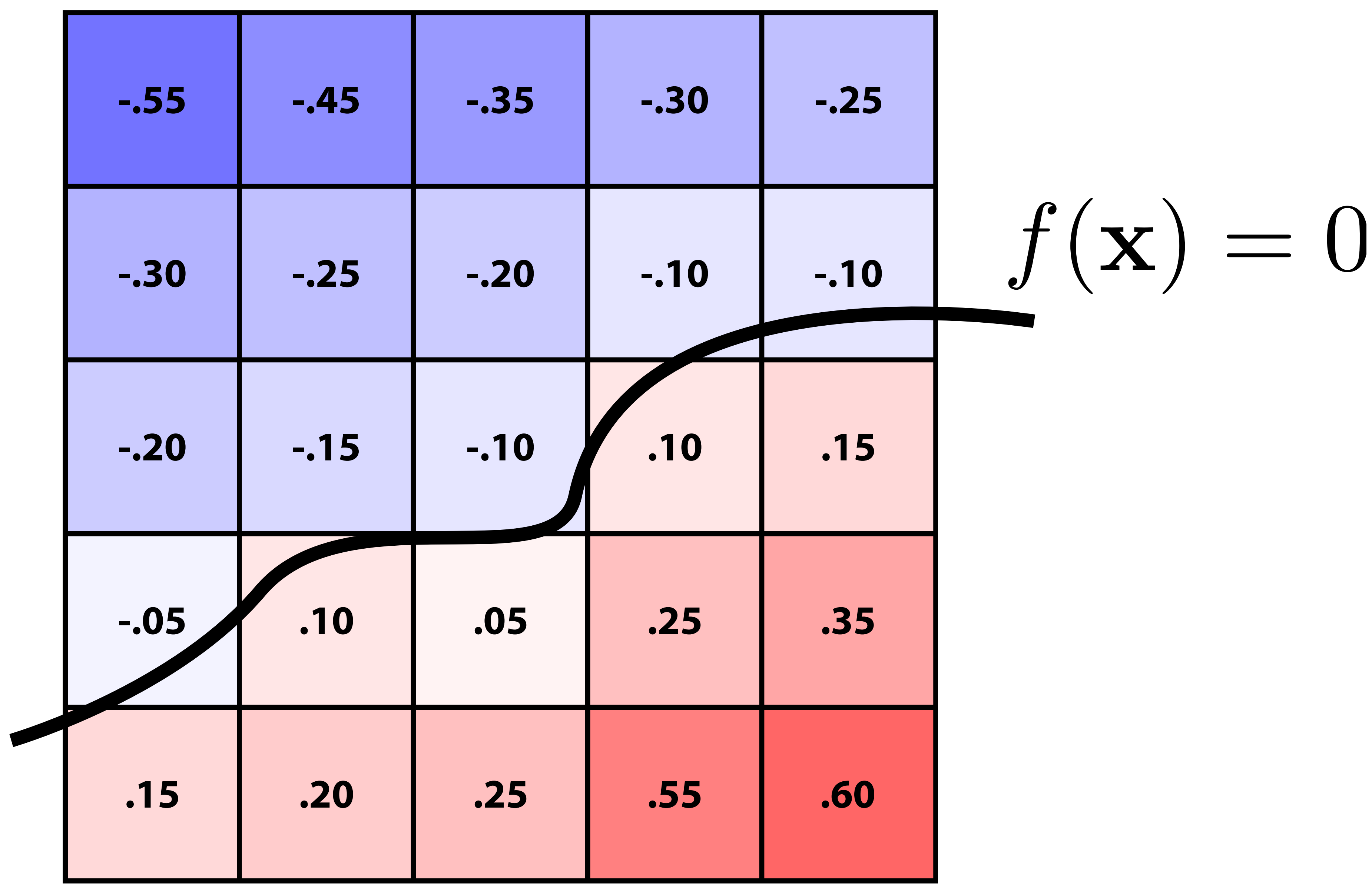
- Appearance depends on exactly how we combine functions
- Q: How do we implement a simple Boolean union?
- A: Just take the minimum: $f(x) = min(d_1(x), d_2(x))$

Scene of pure distance functions (not easy!)

See http://iquilezles.org/www/material/nvscene2008/nvscene2008.htm

Level Set Methods (Implicit)

- Implicit surfaces have some nice features (e.g., merging/splitting)
- But, hard to describe complex shapes in closed form
- Alternative: store a grid of values approximating function



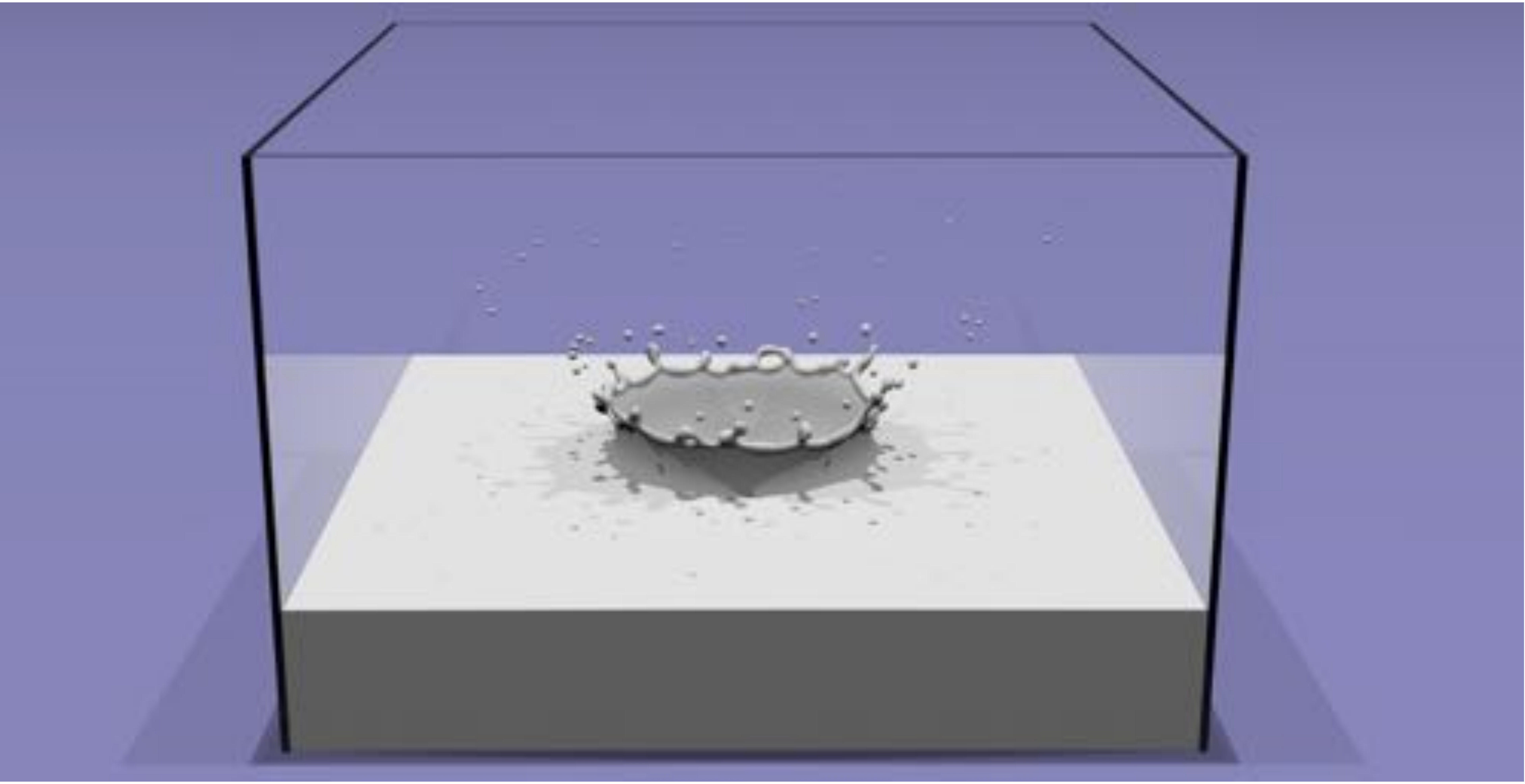
- Surface is found where interpolated values equal zero

Level Sets from Medical Data (CT, MRI, etc.)

- Level sets encode, e.g., constant tissue density

Level Sets in Physical Simulation

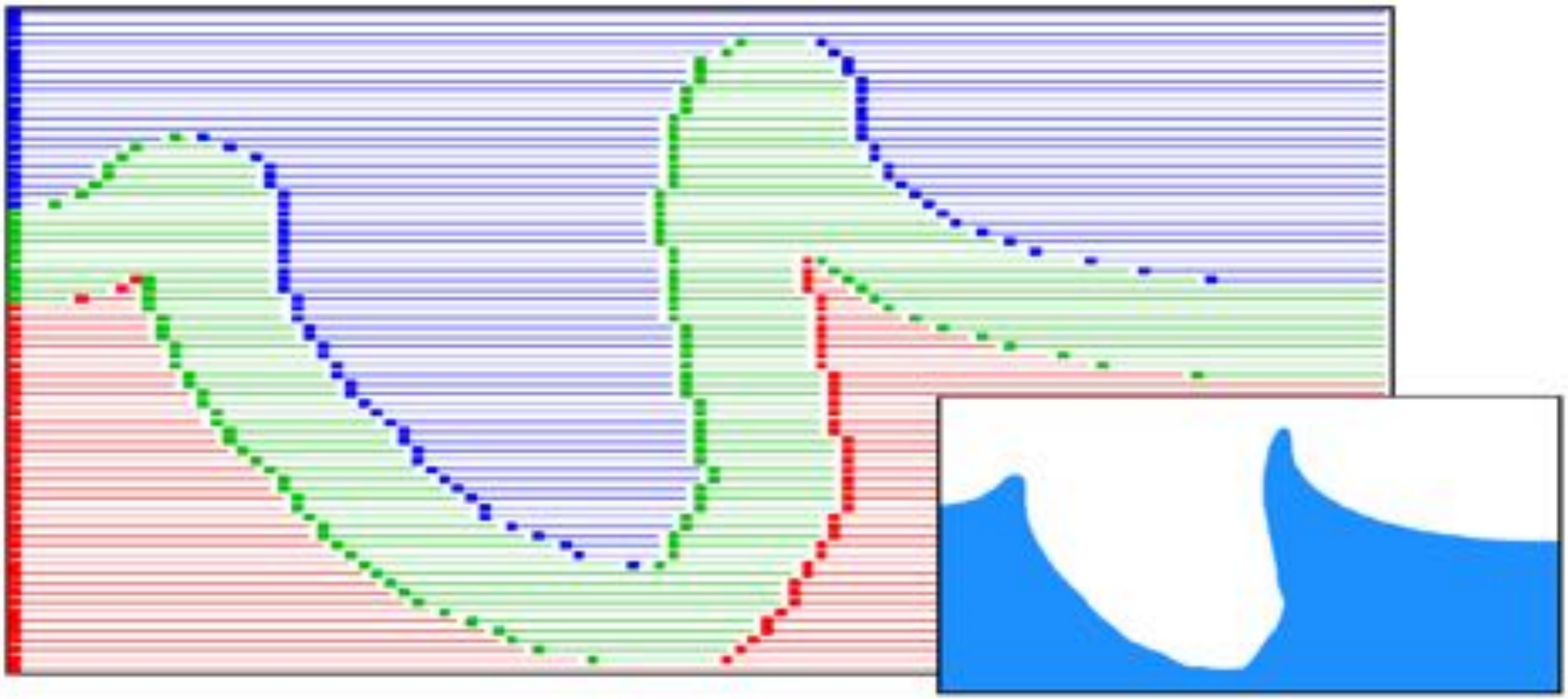
- **Level set encodes distance to air-liquid boundary**



See http://physbam.stanford.edu

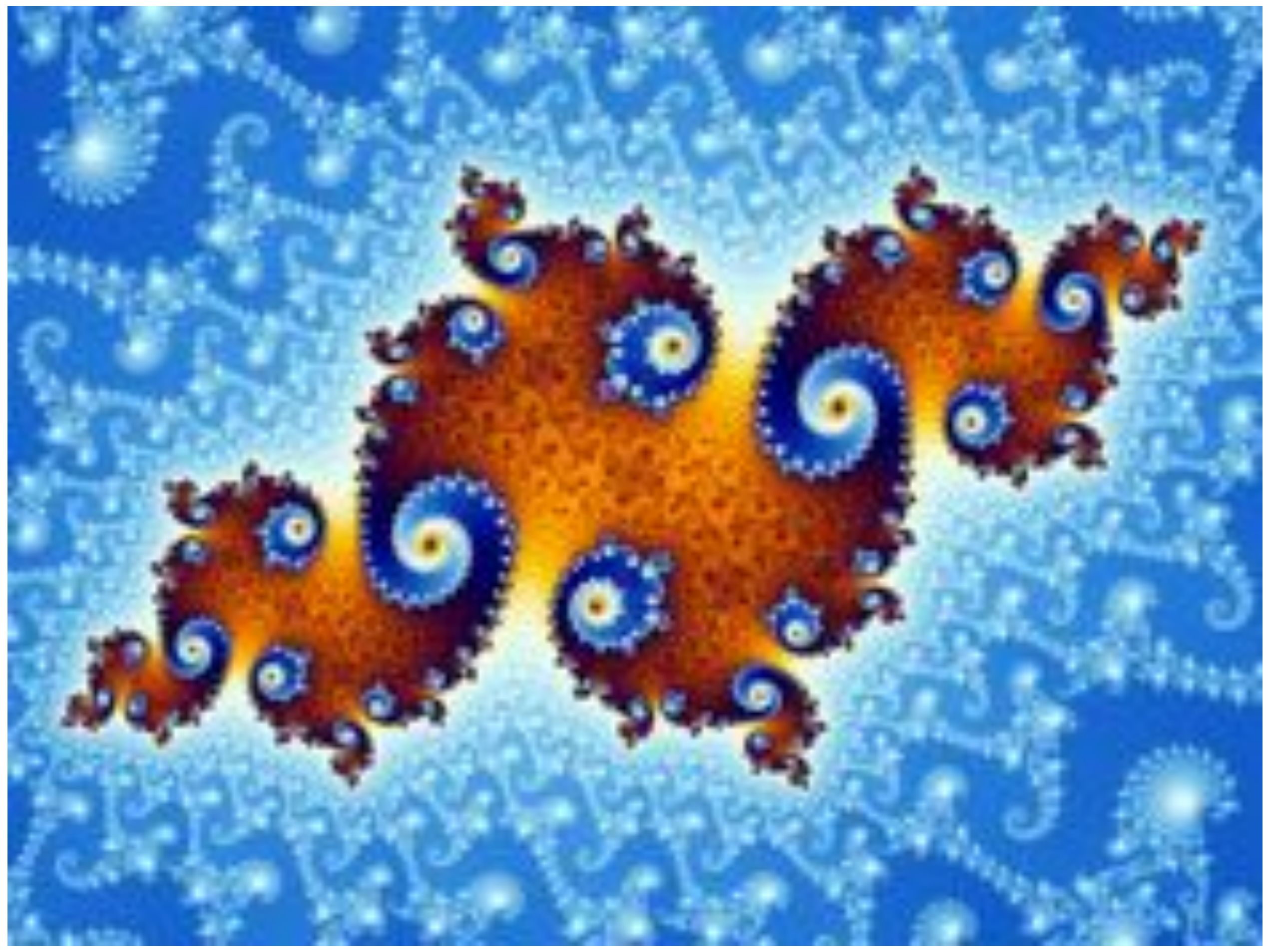
Level Set Storage

- Drawback: storage for 2D surface is now $O(n^3)$
- Can reduce cost by storing only a narrow band around surface:



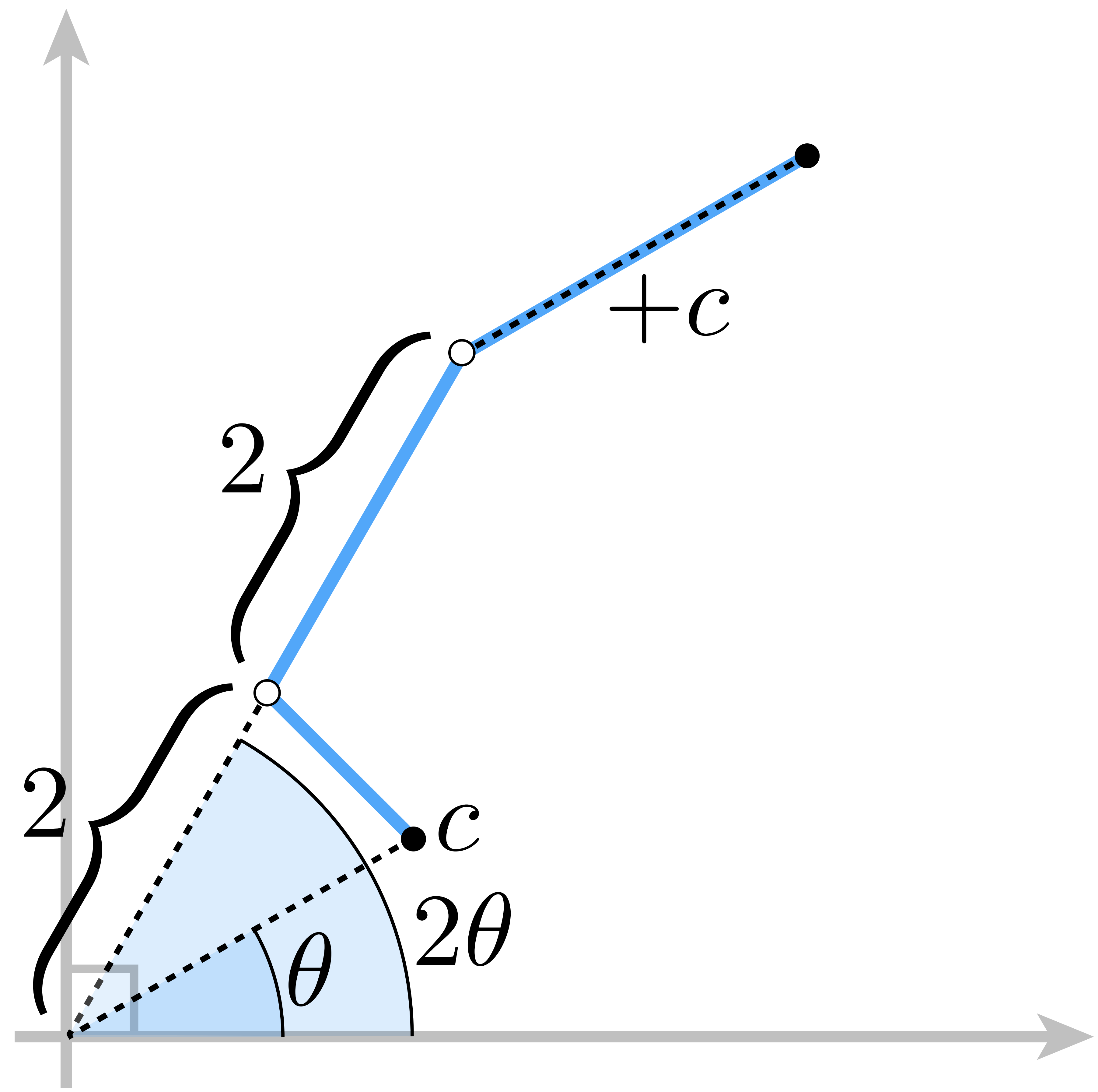
Fractals (Implicit)

- **No precise definition; exhibit self-similarity, detail at all scales**
- **New "language" for describing natural phenomena**
- **Hard to control shape!**



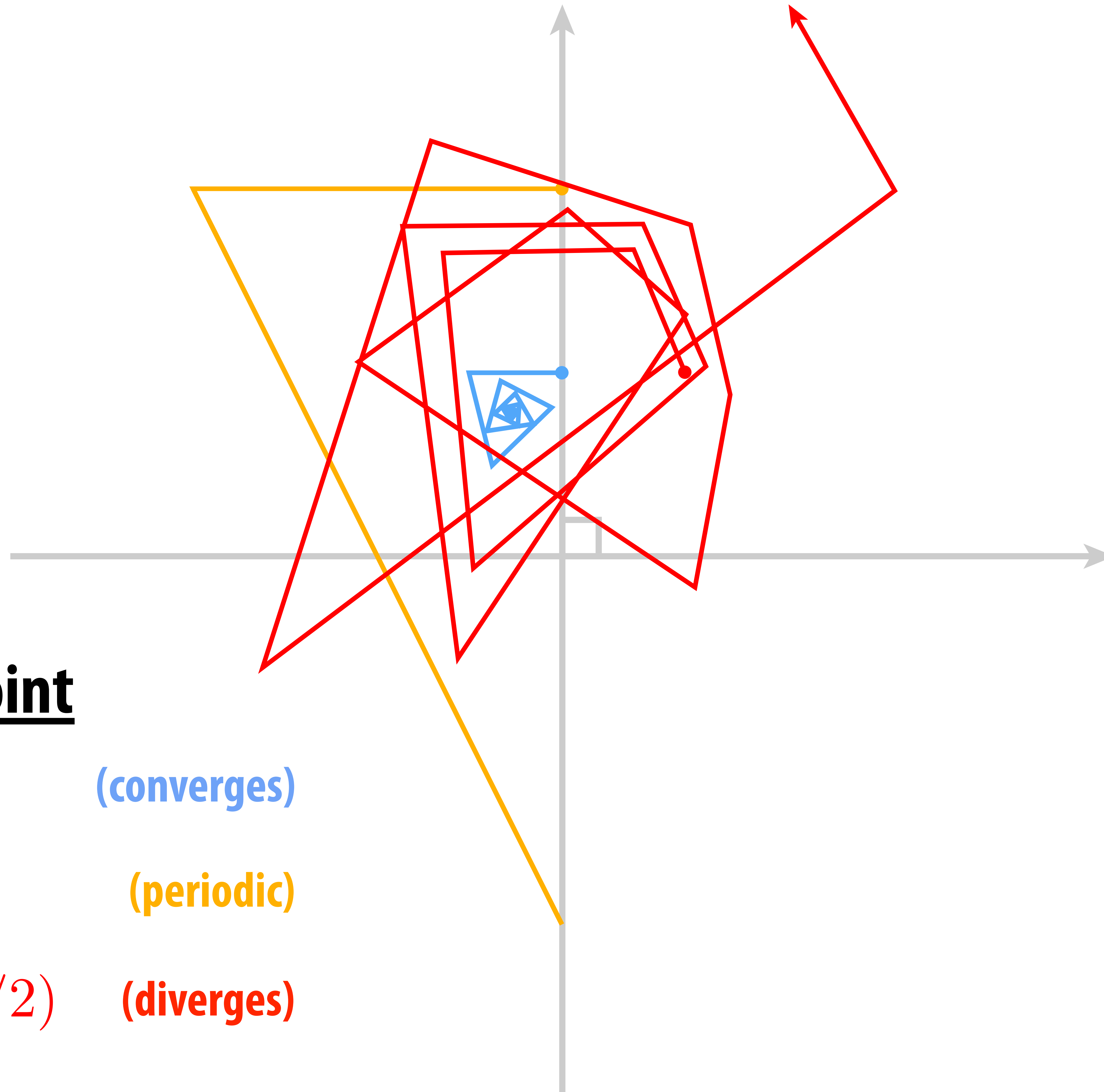
Mandelbrot Set - Definition

- For each point c in the plane:
  - double the angle
  - square the magnitude
  - add the original point c
  - repeat



If the point remains bounded (never goes to ∞), it's in the set.

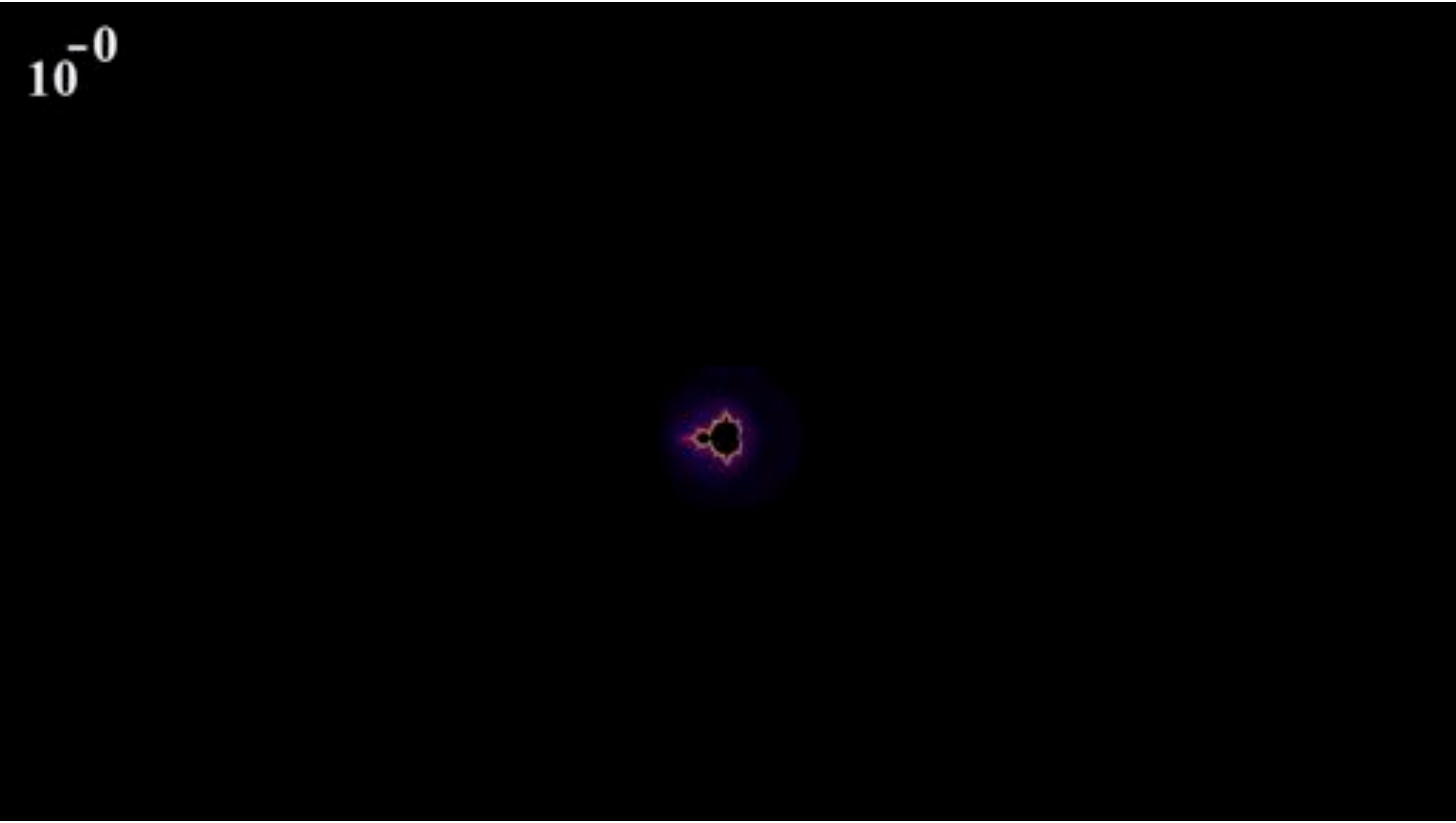
Mandelbrot Set - Examples



starting point

- $(0, 1/2)$ **(converges)**
- $(0, 1)$ **(periodic)**
- $(1/3, 1/2)$ **(diverges)**

Mandelbrot Set - Zooming In



(Colored according to how quickly each point diverges/converges.)

Iterated Function Systems

Scott Draves (CMU Alumn) - see http://electricsheep.org

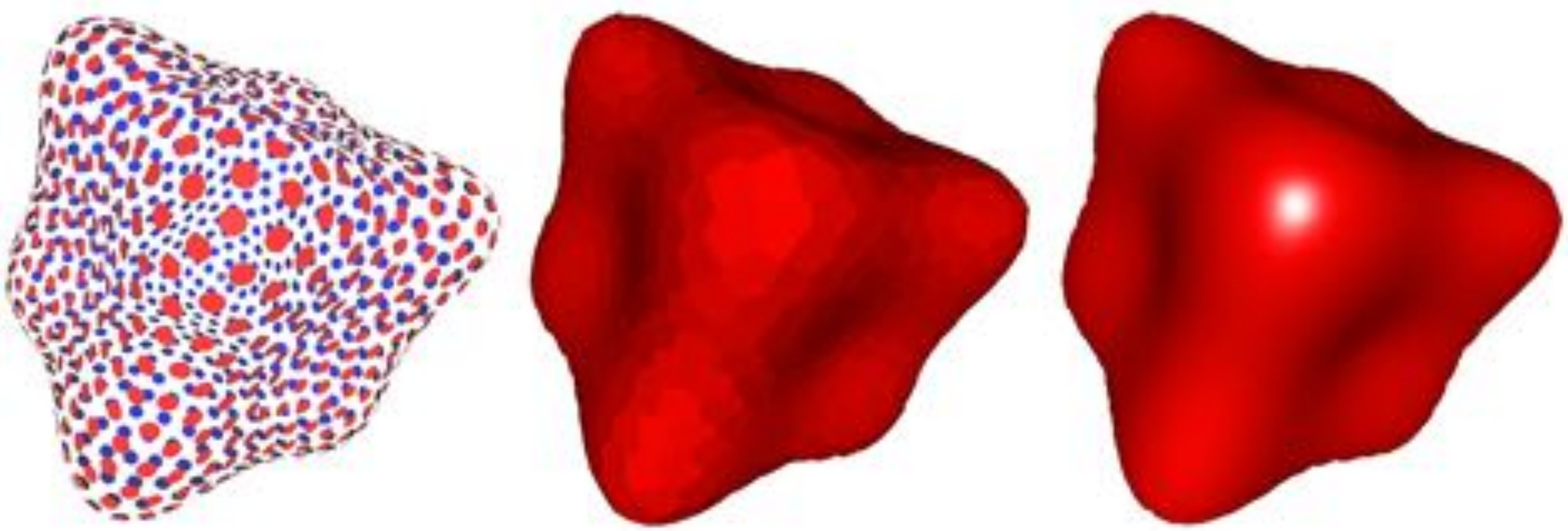
Implicit Representations - Pros & Cons

- **Pros:**
  - description can be very compact (e.g., a polynomial)
  - easy to determine if a point is in our shape (just plug it in!)
  - other queries may also be easy (e.g., distance to surface)
  - for simple shapes, exact description/no sampling error
  - easy to handle changes in topology (e.g., fluid)
- **Cons:**
  - expensive to find all points in the shape (e.g., for drawing)
  - very difficult to model complex shapes

What about explicit representations?

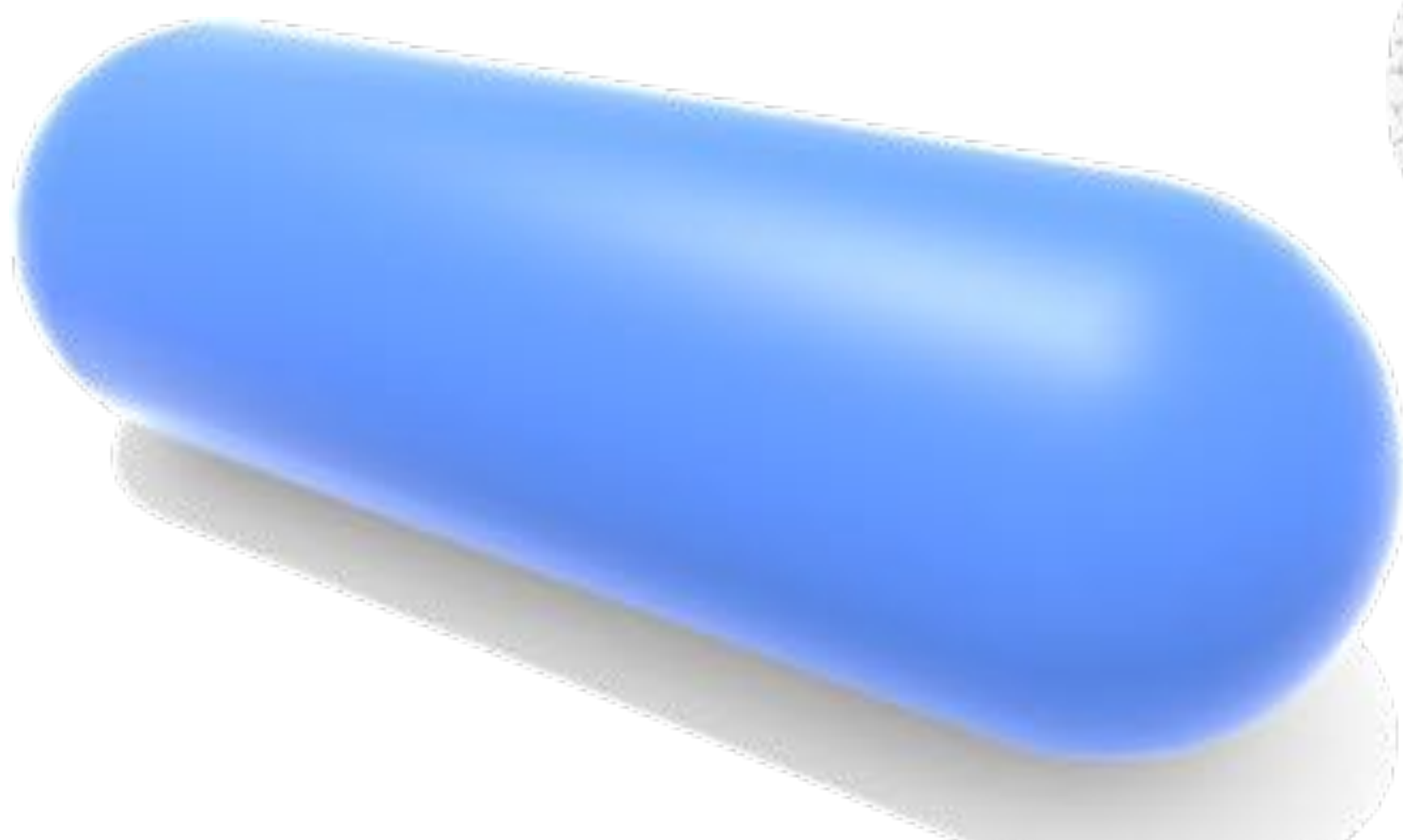
Point Cloud (Explicit)

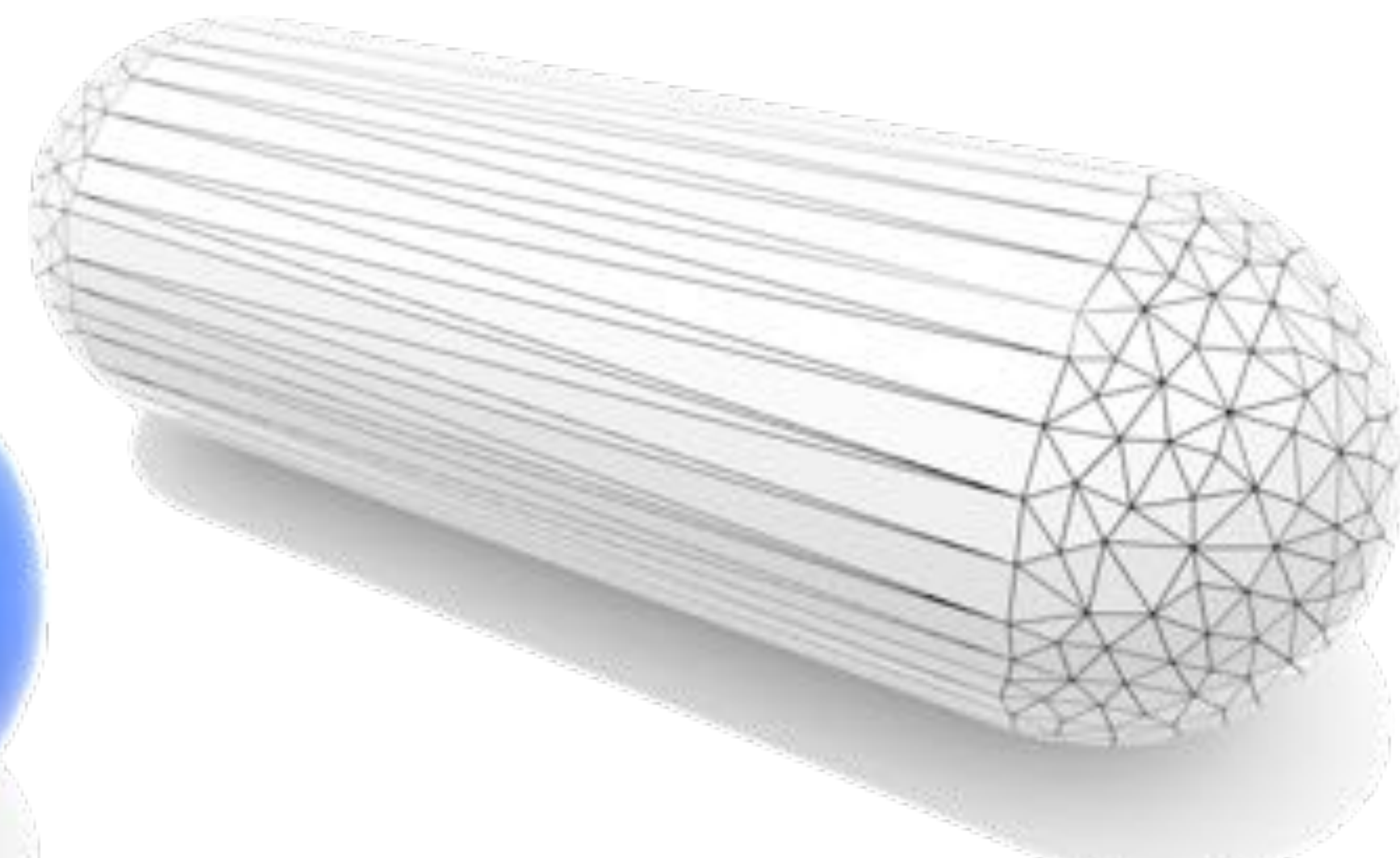
- Easiest representation: list of points (x,y,z)
- Often augmented with normals
- Easily represent any kind of geometry
- Useful for LARGE datasets (>>1 point/pixel)
- Difficult to draw in undersampled regions
- Hard to do processing / simulation



Polygon Mesh (Explicit)

- Store vertices and polygons (most often triangles or quads)
- Easier to do processing/simulation, adaptive sampling
- More complicated data structures
- Perhaps most common representation in graphics



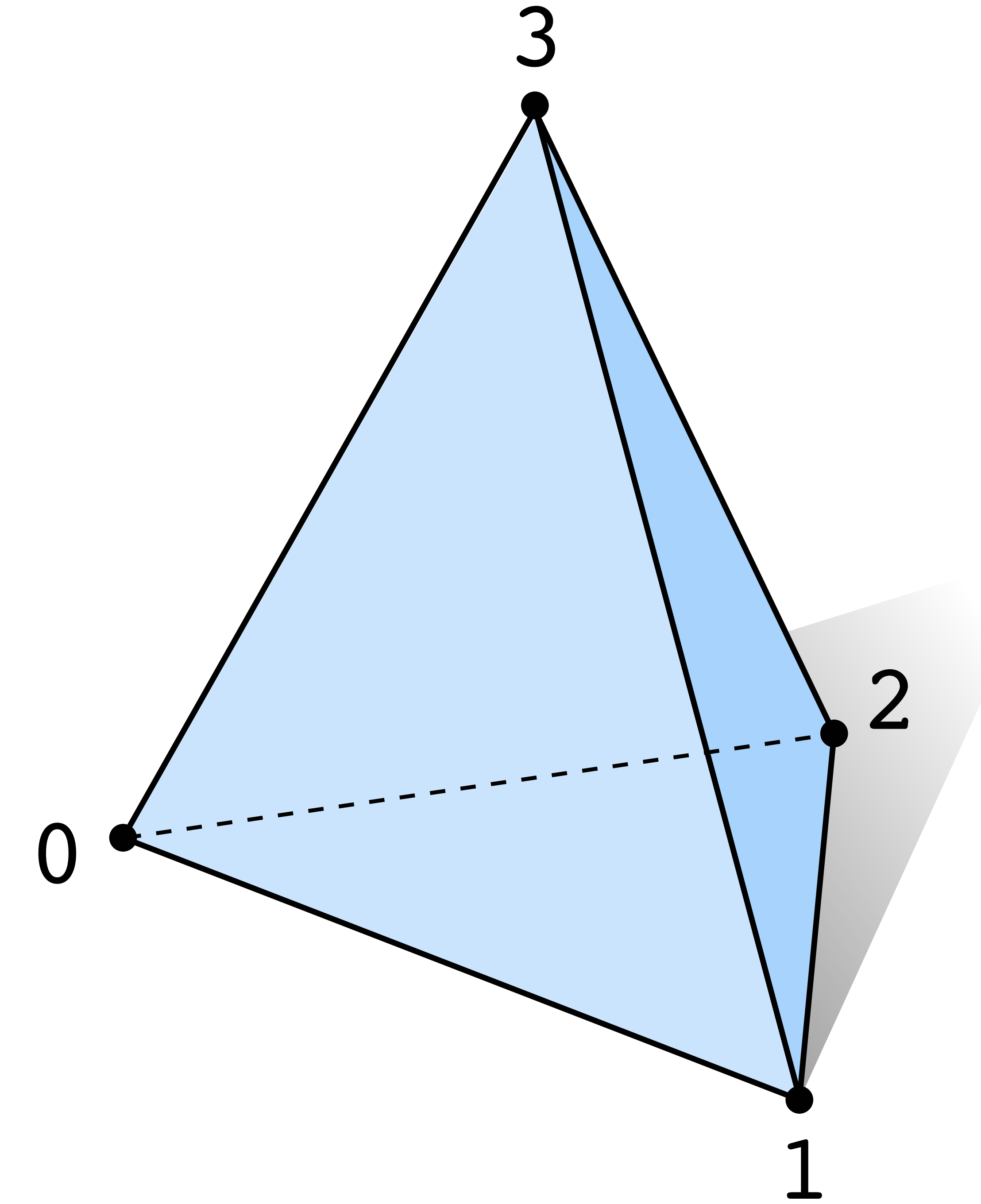


(Much more about polygon meshes in upcoming lectures!)

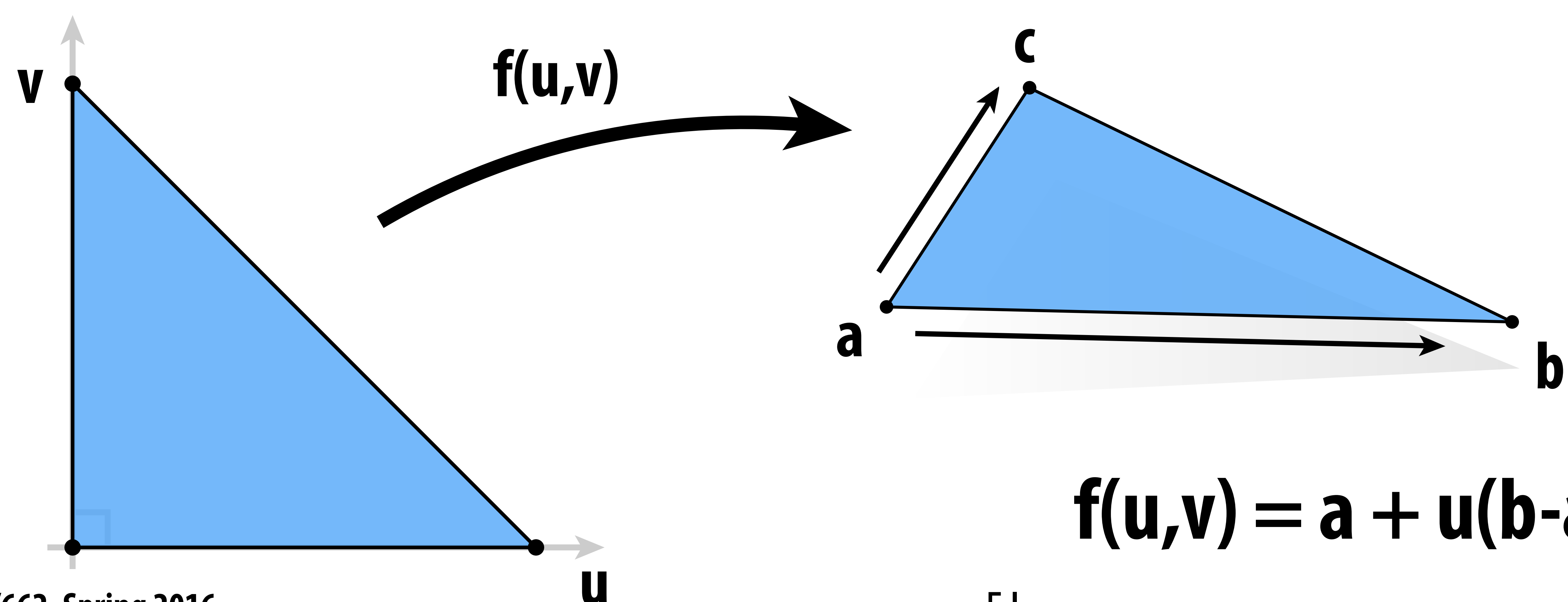
Triangle Mesh (Explicit)

- **Store vertices as triples of coordinates (x,y,z)**
- **Store triangles as triples of indices (i,j,k)**
- **E.g., tetrahedron:**

	VERTICES			TRIANGLES		
	x	y	z	i	j	k
0:	−1	−1	−1	0	2	1
1:	1	−1	1	0	3	2
2:	1	1	−1	3	0	1
3:	−1	1	1	3	1	2



- **Can think of triangle as affine map from plane into space:**



$$f(u,v) = a + u(b-a) + v(c-a)$$

Bernstein Basis

- **Why limit ourselves to just affine functions?**
- **More flexibility by using higher-order polynomials**
- **Instead of usual basis (1, x, x^2, x^3, ...), use Bernstein basis:**

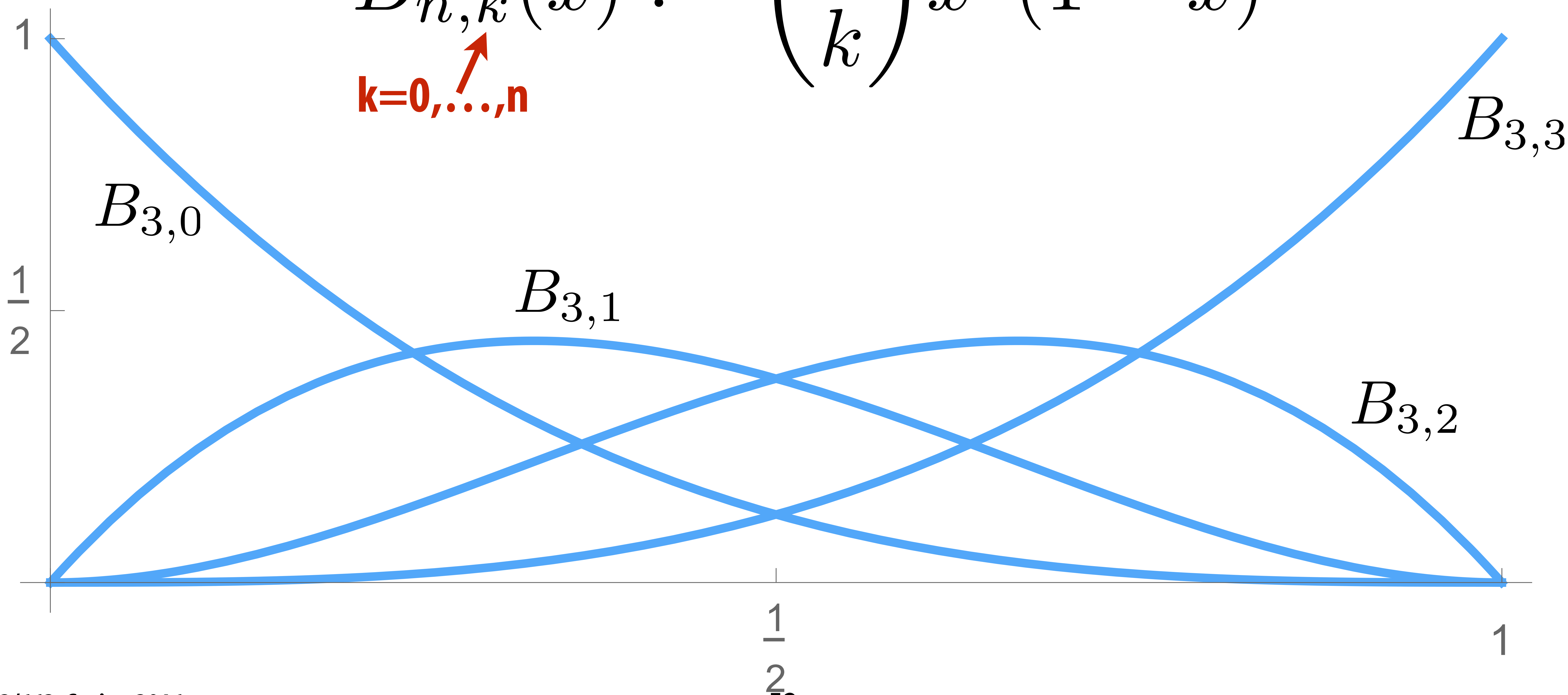
degree

$0 \leq x \leq 1$

"n choose k"

$$B_{n,k}(x) := \binom{n}{k} x^k (1-x)^{n-k}$$

k=0,...,n



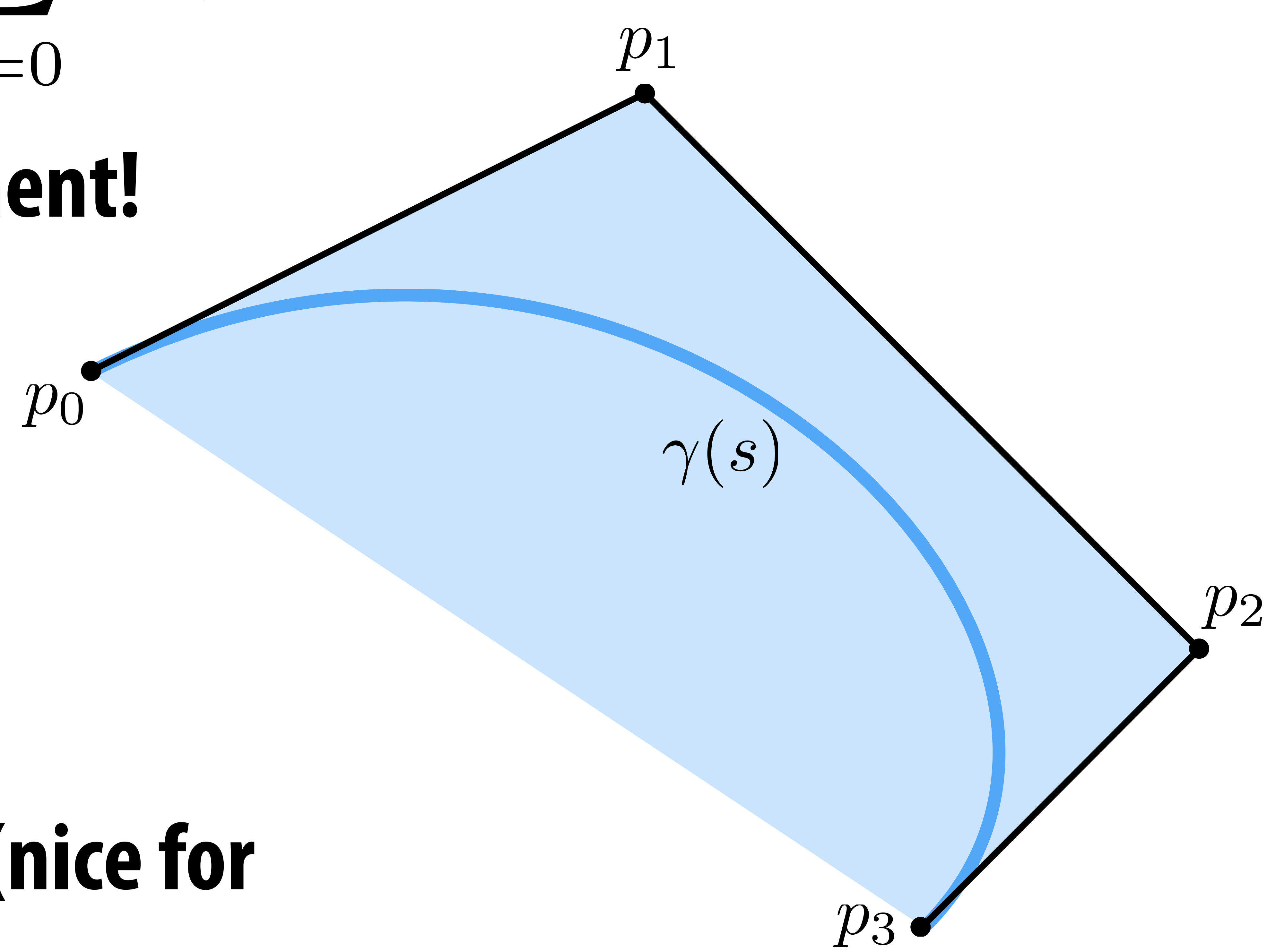
Bézier Curves (Explicit)

- **A Bézier curve is a curve expressed in the Bernstein basis:**

$$\gamma(s) := \sum_{k=0}^{n} B_{n,k}(s)\, p_k$$

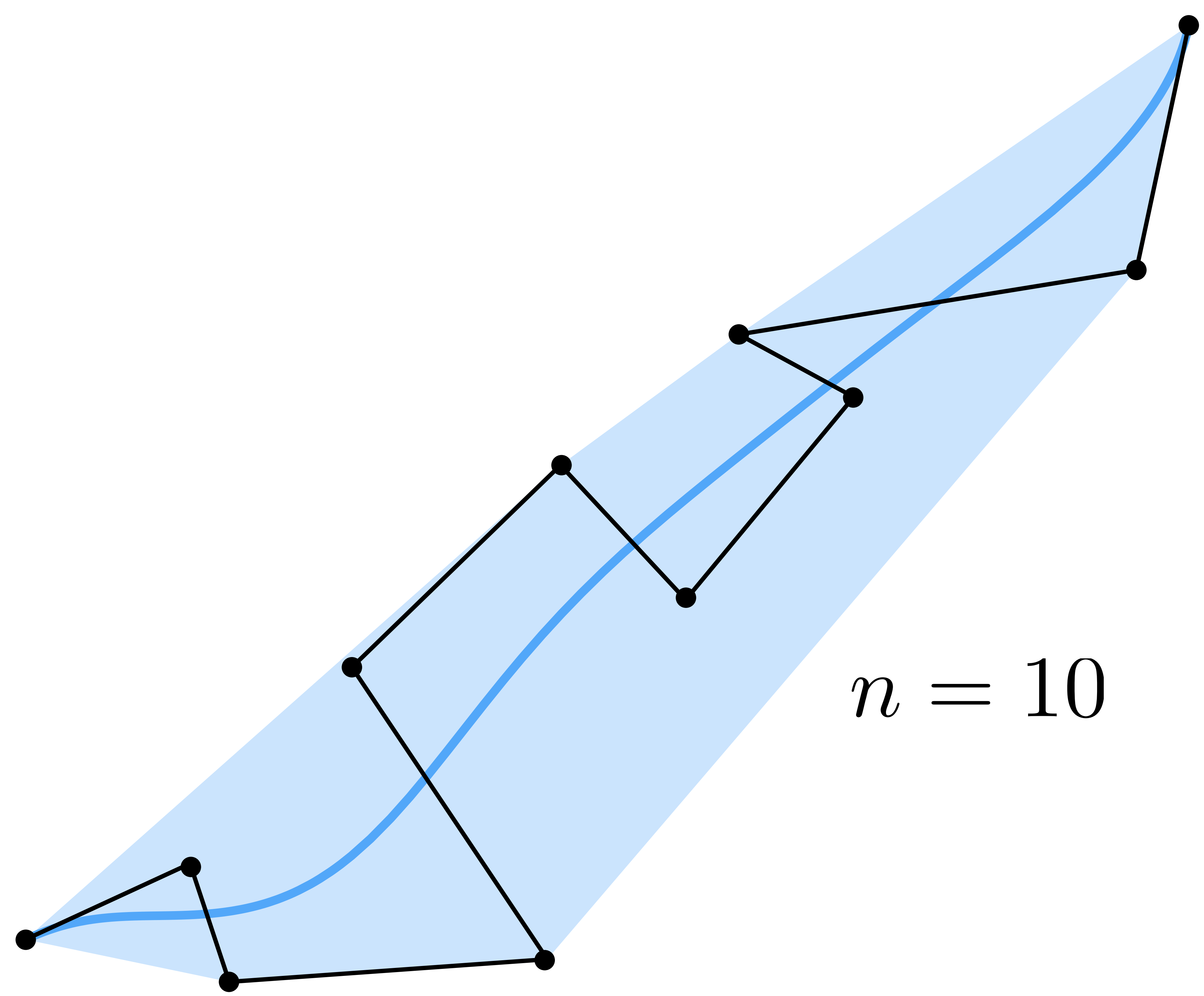
control points

- **For n=1, just get a line segment!**
- **For n=3, get "cubic Bézier":**
- **Important features:**
  1. **interpolates endpoints**
  2. **tangent to end segments**
  3. **contained in convex hull (nice for rasterization)**



Higher-order Bézier Curves?

- What if we want a more interesting curve?
- High-degree Bernstein polynomials don't interpolate well:



Very hard to control!

B-Spline Curves (Explicit)

- Instead, use many low-order Bézier curve (B-spline)
- Widely-used technique in software (Illustrator, Inkscape, etc.)

- Formally, piecewise Bézier curve:

B-spline

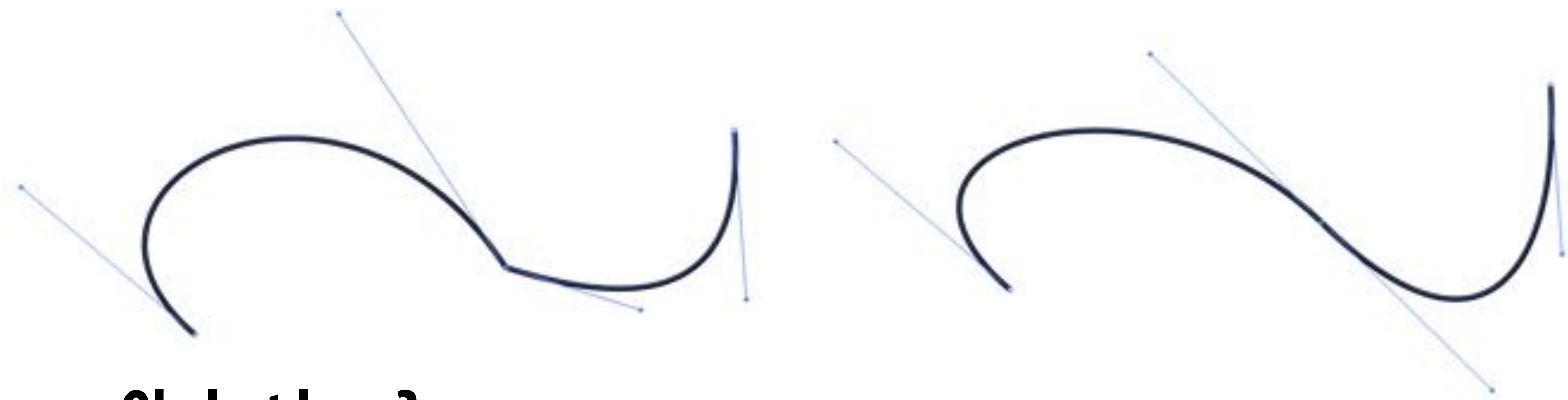
$$\gamma(u) := \gamma_i\left(\frac{u-u_i}{u_{i+1}-u_i}\right), \qquad u_i \le u < u_{i+1}$$

Bézier

- Location of u_i parameters are called "knots"

B-Splines — tangent continuity

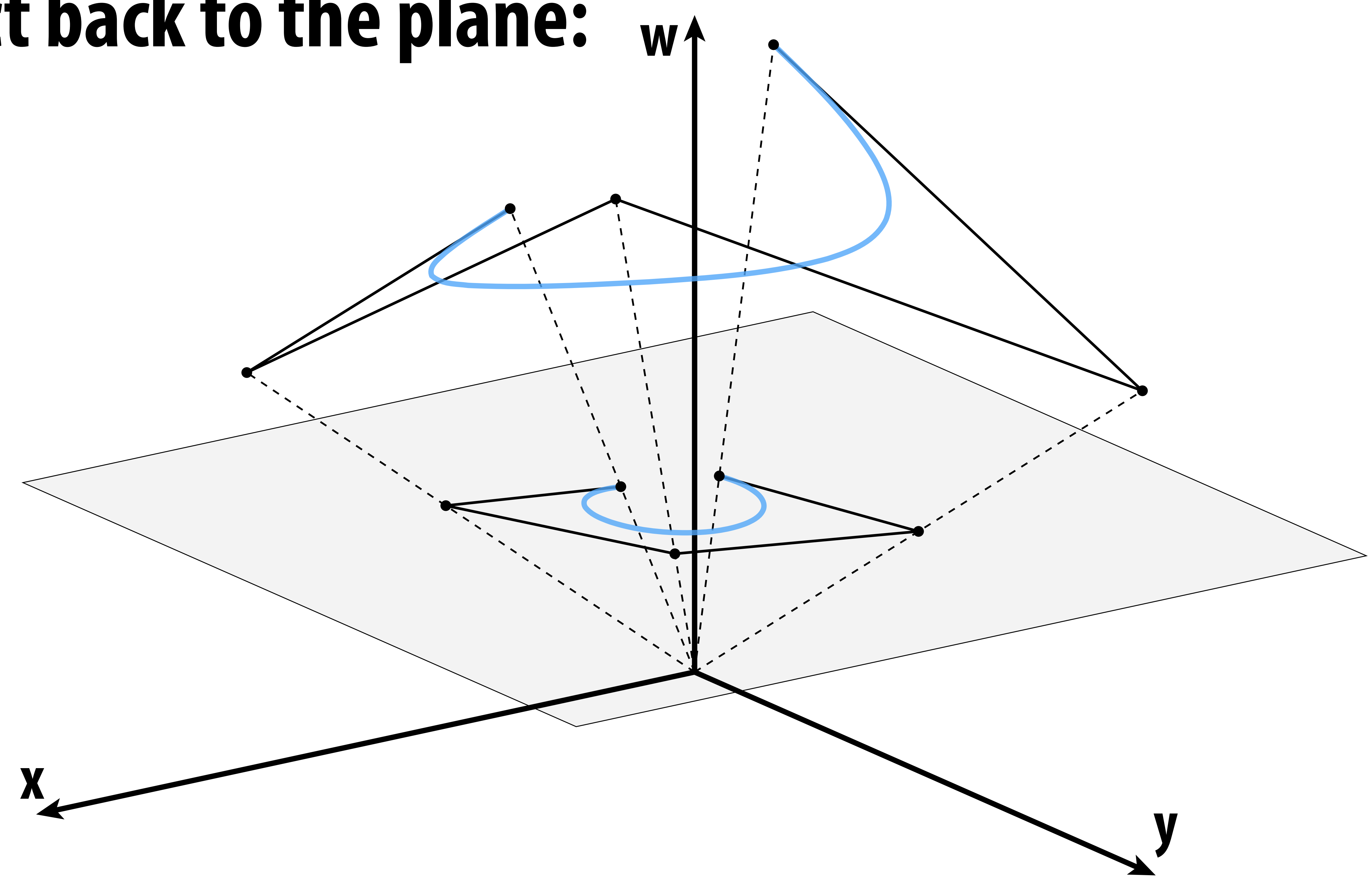
- **To get "seamless" curves, want tangents to line up:**



- **Ok, but how?**
- **Each curve is cubic: $u^3p_0 + 3u^2(1-u)p_1 + 3u(1-u)^2p_2 + (1-u)^3p_3$**
- **Q: How many degrees of freedom in a single cubic Bézier**
- **Tangents are difference between first two & last two points**
- **Q: How many degrees of freedom per B-spline segment?**
- **Q: Could you do this with quadratic Bézier? Linear Bézier?**

Rational B-Splines (Explicit)

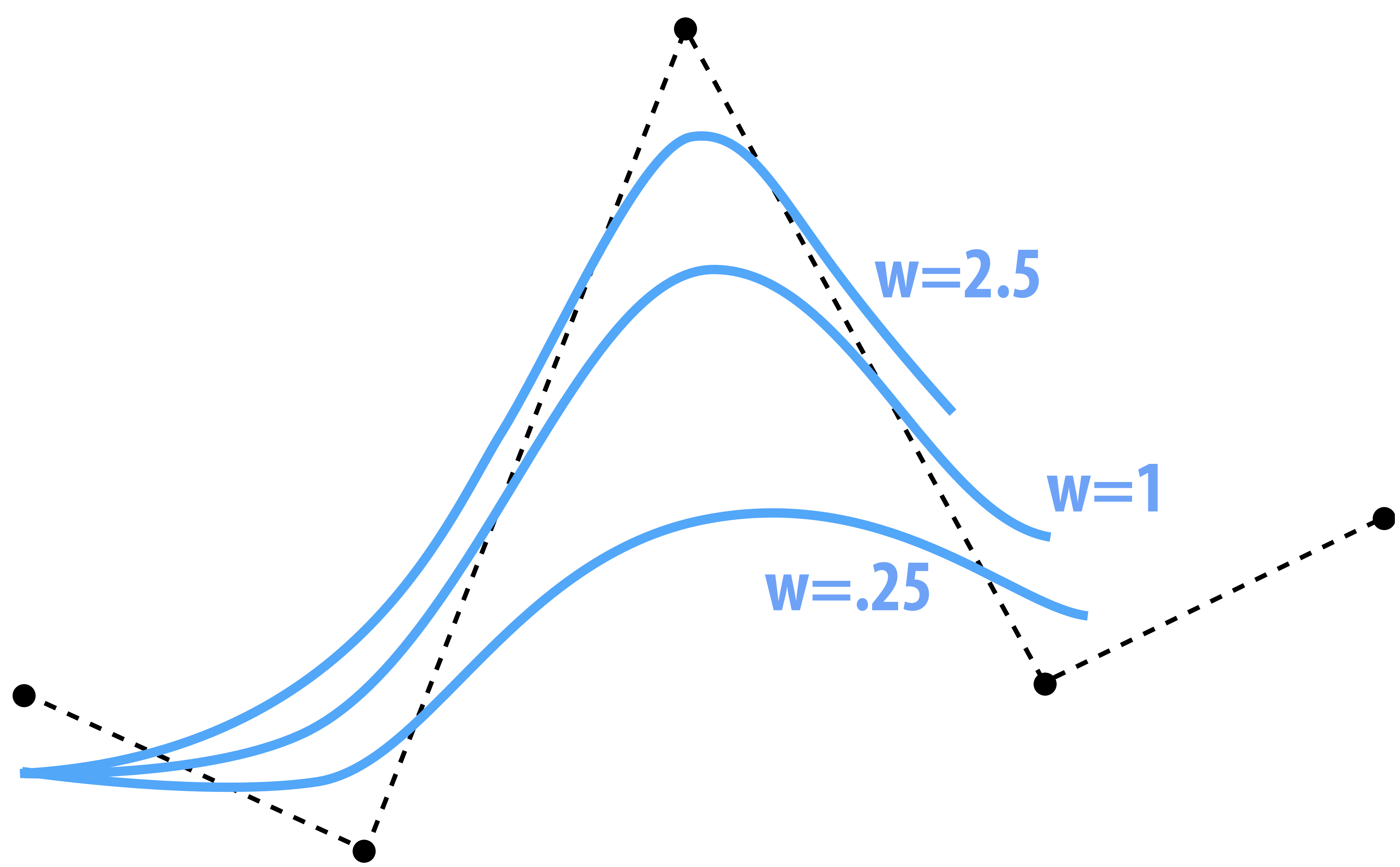
- B-Splines can't exactly represent conics—not even the circle!
- Solution: interpolate in homogeneous coordinates, then project back to the plane:



Result is called a rational B-spline.

NURBS (Explicit)

- **(N)on-(U)niform (R)ational (B)-(S)pline**
  - **knots at arbitrary locations (non-uniform)**
  - **expressed in homogeneous coordinates (rational)**
  - **piecewise polynomial curve (B-Spline)**
- **Homogeneous coordinate w controls "strength" of a vertex:**

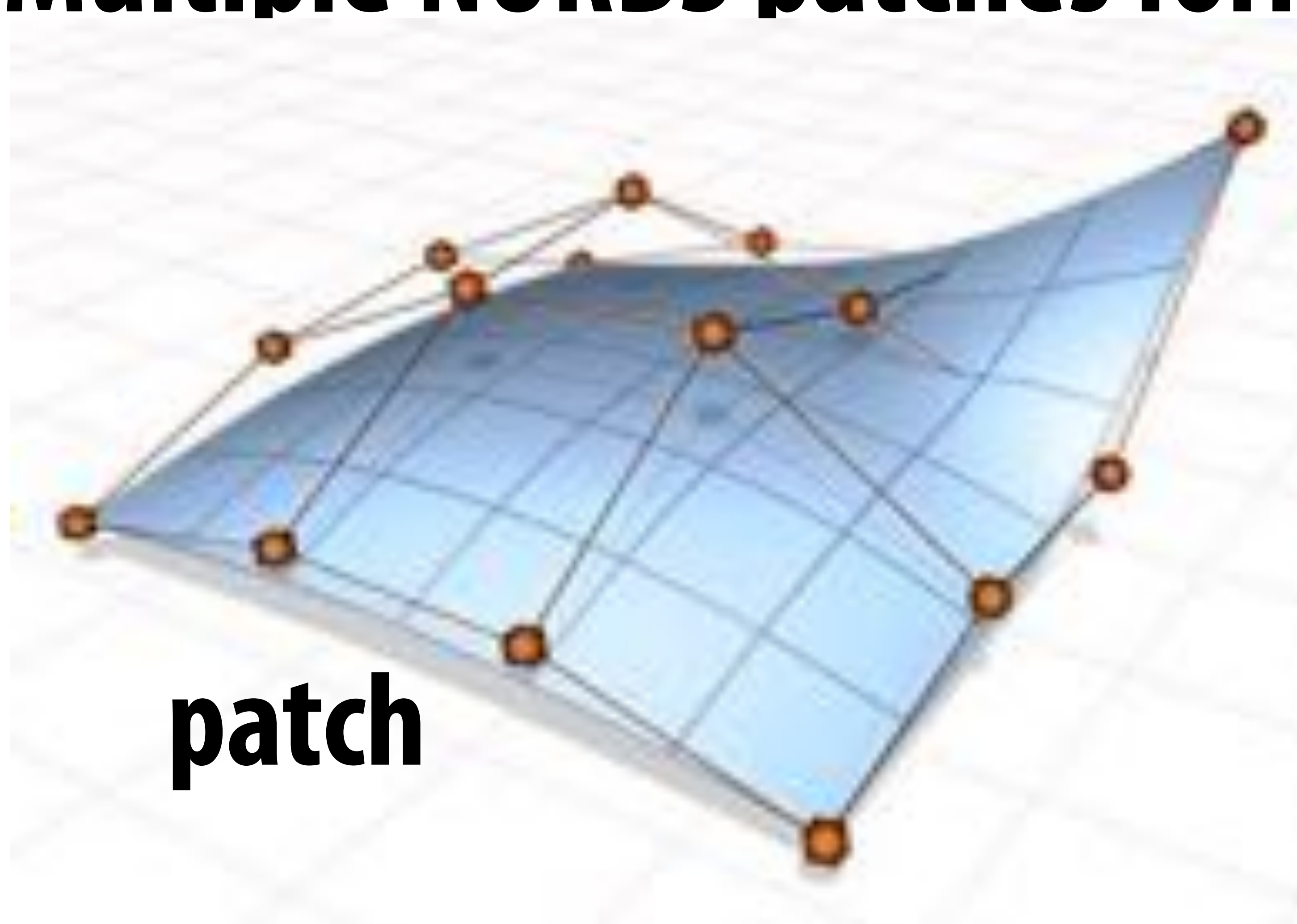


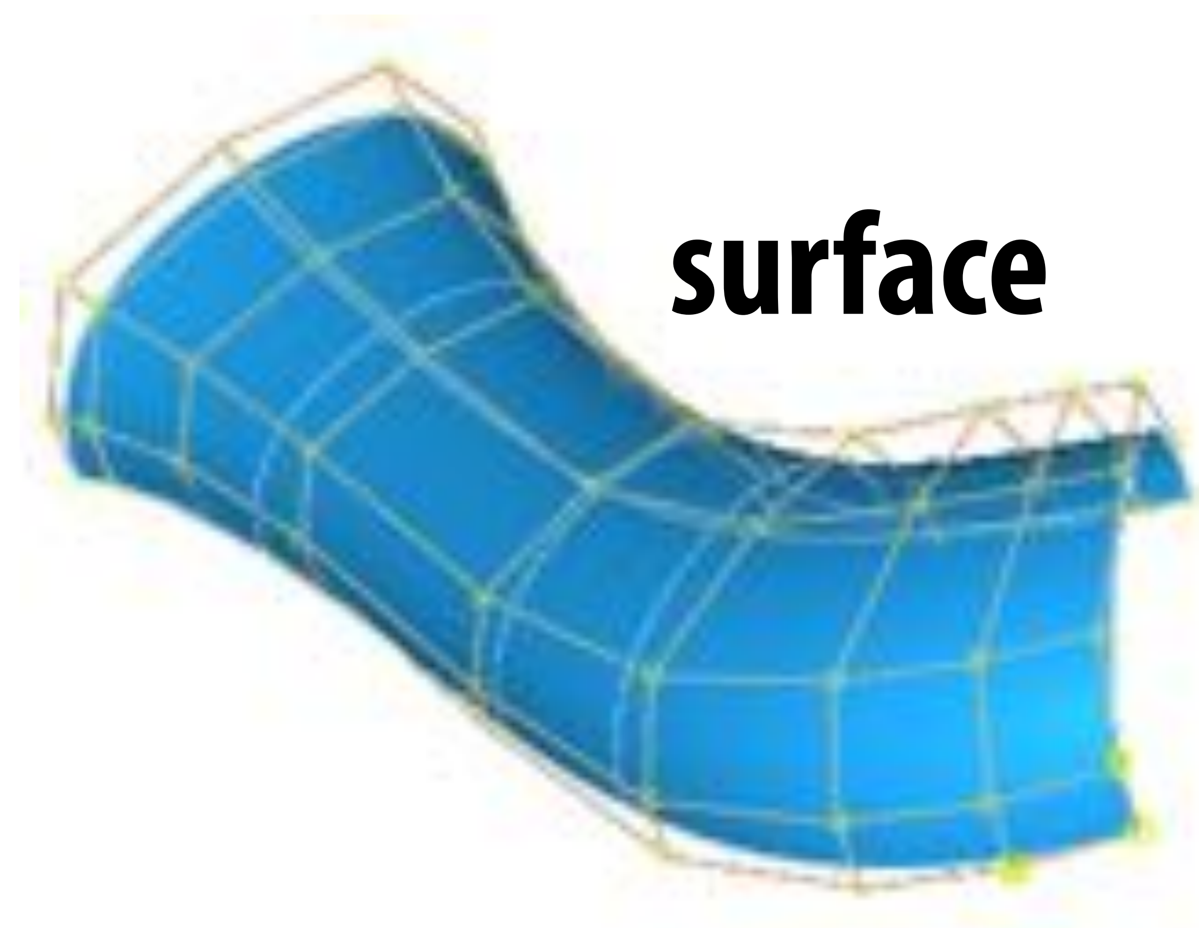
NURBS Surface (Explicit)

- Much more common than using NURBS for curves
- Use tensor product of scalar NURBS curves to get a patch:

$$S(u, v) := N_i(u) N_j(v) p_{ij}$$

- Multiple NURBS patches form a surface





- Pros: easy to evaluate, exact conics, high degree of continuity
- Cons: Hard to piece together patches, hard to edit (many DOFs)

What you should know:

- List some types of implicit surface representations
- What types of operations are easy with implicit surface representations?
- List some types of explicit surface representations
- What types of operations are easy with explicit surface representations?
- What is CSG (constructive solid geometry)?   Give some examples of CSG operations.
- What type of representation is best for CSG operations?
- Describe how to do union, intersection, and subtraction of geometry using simple operators on a surface representation.
- What is a level set representation?   When is it useful?
- What types of splines are common in computer graphics?
- Why are they popular?   What properties make them most useful?
- Derive the equation on slide 56 from the definitions on slides 52 and 53.   Draw a diagram to illustrate any terms you use.
- What was one motivation behind developing the rational b-spline?
- What is one advantage of using a *non-uniform* rational b-spline?