Parametric Curves

Modeling:
- parametric curves (Splines)
- polygonal meshes
Modeling Complex Shapes

- We want to build models of very complicated objects
- An equation for a sphere is possible, but how about an equation for a telephone, or a face?
- Complexity is achieved using simple pieces
  - polygons, parametric curves and surfaces, or implicit curves and surfaces
  - This lecture: parametric curves
What Do We Need From Curves in Computer Graphics?

- Local control of shape (so that easy to build and modify)
- Stability
- Smoothness and continuity
- Ability to evaluate derivatives
- Ease of rendering
Curve Representations

- Explicit: $y = f(x)$
  
  $y = mx + b$
  
  - Easy to generate points
  - Must be a function: big limitation—vertical lines?
Curve Representations

- **Explicit**: \( y = f(x) \)
  
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- **Implicit**: \( f(x, y) = 0 \)
  
  \[ x^2 + y^2 - r^2 = 0 \]
  
  + Easy to test if on the curve
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• Parametric: $(x,y) = (f(u), g(u))$
  
  $$(x, y) = (\cos u, \sin u)$$
  
  + Easy to generate points
Parameterization of a Curve

- *Parameterization* of a curve: how a change in $u$ moves you along a given curve in $xyz$ space.
Polynomial Interpolation

• An $n$-th degree polynomial fits a curve to $n+1$ points
  – called Lagrange Interpolation
  – result is a curve that is too wiggly, change to any control point affects entire curve (nonlocal) – *this method is poor*

• We usually want the curve to be as smooth as possible
  – minimize the wiggles
  – high-degree polynomials are bad
Splines: Piecewise Polynomials

- A spline is a *piecewise polynomial* - many low degree polynomials are used to interpolate (pass through) the control points.

- *Cubic piecewise* polynomials are the most common:
  - piecewise definition gives local control.
Piecewise Polynomials

- Spline: lots of little polynomials pieced together
- Want to make sure they fit together nicely

- \( C_0 \) continuity: Continuous in position
- \( C_0 \) and \( C_1 \) continuity: Continuous in position and tangent vector
- \( C_0 \) and \( C_1 \) and \( C_2 \) continuity: Continuous in position, tangent, and curvature
Splines

- Types of splines:
  - Hermite Splines
  - Catmull-Rom Splines
  - Bezier Splines
  - Natural Cubic Splines
  - B-Splines
  - NURBS
Hermite Curves

• Cubic Hermite Splines

That is, we want a way to specify the end points and the slope at the end points!
Splines

chalkboard
The Cubic Hermite Spline Equation

• Using some algebra, we obtain:

\[ p(u) = \begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ \nabla p_1 \\ \nabla p_2 \end{bmatrix} \]

point that gets drawn

basis (geometry) matrix

control matrix (what the user gets to pick)

• This form typical for splines
  – basis matrix and meaning of control matrix change with the spline type
The Cubic Hermite Spline Equation

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\[ p(u) = \begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ \nabla p_1 \\ \nabla p_2 \end{bmatrix} \]

Point that gets drawn

Basis

Control matrix (what the user gets to pick)

\[ p(u) = \begin{bmatrix} 2u^3 - 3u^2 + 1 \\ -2u^3 + 3u^2 \\ u^3 - 2u^2 + u \\ u^3 - u^2 \\ 15 \end{bmatrix}^T \begin{bmatrix} p_1 \\ p_2 \\ \nabla p_1 \\ \nabla p_2 \end{bmatrix} \]

4 Basis Functions
Four Basis Functions for Hermite splines

\[ p(u) = \begin{bmatrix}
2u^3 - 3u^2 + 1 \\
-2u^3 + 3u^2 \\
u^3 - 2u^2 + u \\
u^3 - u^2
\end{bmatrix} \begin{bmatrix}
p_1 \\
p_2 \\
\nabla p_1 \\
\nabla p_2
\end{bmatrix} \]

Hermite Blending Functions

Every cubic Hermite spline is a linear combination (blend) of these 4 functions
Piecing together Hermite Curves

- It's easy to make a multi-segment Hermite spline
  - each piece is specified by a cubic Hermite curve
  - just specify the position and tangent at each “joint”
  - the pieces fit together with matched positions and first derivatives
  - gives C1 continuity

- The points that the curve has to pass through are called *knots* or *knot points*
Catmull-Rom Splines

• With Hermite splines, the designer must specify all the tangent vectors
• Catmull-Rom: an interpolating cubic spline with built-in $C^1$ continuity.
Catmull-Rom Splines

- With Hermite splines, the designer must specify all the tangent vectors.
- Catmull-Rom: an interpolating cubic spline with built-in $C^1$ continuity.

$tangent \text{ at } p_i = s(p_{i+1} - p_{i-1})$
Catmull-Rom Spline Matrix

\[ p(u) = \begin{bmatrix} u^3 & u^2 & u \\ \end{bmatrix} \begin{bmatrix} -s & 2-s & s-2 & s \\ 2s & s-3 & 3-2s & -s \\ -s & 0 & s & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix} \]

- Derived similarly to Hermite
- Parameter \( s \) is typically set to \( s=1/2 \).
Cubic Curves in 3D

- Three cubic polynomials, one for each coordinate
  - \( x(u) = a_x u^3 + b_x u^2 + c_x u + d_x \)
  - \( y(u) = a_y u^3 + b_y u^2 + c_y u + d_y \)
  - \( z(u) = a_z u^3 + b_z u^2 + c_z u + d_z \)

- In matrix notation

\[
\begin{bmatrix}
  x(u) & y(u) & z(u)
\end{bmatrix} = \begin{bmatrix}
  u^3 & u^2 & u & 1
\end{bmatrix} \begin{bmatrix}
  a_x & a_y & a_z \\
  b_x & b_y & b_z \\
  c_x & c_y & c_z \\
  d_x & d_y & d_z
\end{bmatrix}
\]
Catmull-Rom Spline Matrix in 3D

\[
\begin{bmatrix}
  x(u) & y(u) & z(u)
\end{bmatrix} = \begin{bmatrix}
  u^3 & u^2 & u & 1
\end{bmatrix} \begin{bmatrix}
  -s & 2-s & s-2 & s \\
  2s & s-3 & 3-2s & -s \\
  -s & 0 & s & 0 \\
  0 & 1 & 0 & 0
\end{bmatrix} \begin{bmatrix}
  x_1 & y_1 & z_1 \\
  x_2 & y_2 & z_2 \\
  x_3 & y_3 & z_3 \\
  x_4 & y_4 & z_4
\end{bmatrix}
\]

CR basis             control vector
Beziers Curves*

- Another variant of the same game
- Instead of endpoints and tangents, four control points
  - points $P_0$ and $P_3$ are on the curve: $P(u=0) = P_0, \quad P(u=1) = P_3$
  - points $P_1$ and $P_2$ are off the curve
  - $P'(u=0) = 3(P_1 - P_0), \quad P'(u=1) = 3(P_3 - P_2)$
- Convex Hull property
  - curve contained within convex hull of control points
- Gives more control knobs (series of points) than Hermite
- Scale factor (3) is chosen to make “velocity” approximately constant
The Bezier Spline Matrix*

\[
\begin{bmatrix}
  x \\
  y \\
  z
\end{bmatrix} =
\begin{bmatrix}
  u^3 & u^2 & u & 1
\end{bmatrix}
\begin{bmatrix}
  -1 & 3 & -3 & 1 \\
  3 & -6 & 3 & 0 \\
  -3 & 3 & 0 & 0 \\
  1 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
  x_1 & y_1 & z_1 \\
  x_2 & y_2 & z_2 \\
  x_3 & y_3 & z_3 \\
  x_4 & y_4 & z_4
\end{bmatrix}
\]

Bezier basis

Bezier control vector
Beziers Blending Functions* 

\[
p(t) = \begin{bmatrix}
(1-t)^3 \\
3t(1-t)^2 \\
3t^2(1-t) \\
t^3
\end{bmatrix}^T \begin{bmatrix}
p_1 \\
p_2 \\
p_3 \\
p_4
\end{bmatrix}
\]

Also known as the order 4, degree 3 Bernstein polynomials. 
Nonnegative, sum to 1. 
The entire curve lies inside the polyhedron bounded by the control points.
Splines with More Continuity?

• How could we get $C^2$ continuity at control points?

• Possible answers:
  – Use higher degree polynomials
    degree 4 = quartic, degree 5 = quintic, … but these get computationally expensive, and sometimes wiggly
  – Give up local control natural cubic splines
    A change to any control point affects the entire curve
  – Give up interpolation cubic B-splines
    Curve goes near, but not through, the control points
Piecewise Polynomials

- Spline: lots of little polynomials pieced together
- Want to make sure they fit together nicely

- $C_0$ continuity: Continuous in position
- $C_0$ & $C_1$ continuity: Continuous in position and tangent vector
- $C_0$ & $C_1$ & $C_2$ continuity: Continuous in position, tangent, and curvature
## Comparison of Basic Cubic Splines

<table>
<thead>
<tr>
<th>Type</th>
<th>Local Control</th>
<th>Continuity</th>
<th>Interpolation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hermite</td>
<td>YES</td>
<td>C1</td>
<td>YES</td>
</tr>
<tr>
<td>Bezier</td>
<td>YES</td>
<td>C1</td>
<td>YES</td>
</tr>
<tr>
<td>Catmull-Rom</td>
<td>YES</td>
<td>C1</td>
<td>YES</td>
</tr>
<tr>
<td>Natural</td>
<td>NO</td>
<td>C2</td>
<td>YES</td>
</tr>
<tr>
<td>B-Splines</td>
<td>YES</td>
<td>C2</td>
<td>NO</td>
</tr>
</tbody>
</table>

- **Summary**
  - Can’t get C2, interpolation and local control with cubics
B-Splines*

- Give up interpolation
  - the curve passes near the control points
  - best generated with interactive placement (because it’s hard to guess where the curve will go)
- Curve obeys the convex hull property
- C2 continuity and local control are good compensation for loss of interpolation
B-Spline Basis

- We always need 3 more control points than spline pieces

\[
M_{Bs} = \frac{1}{6} \begin{bmatrix}
-1 & 3 & -3 & 1 \\
3 & -6 & 3 & 0 \\
-3 & 0 & 3 & 0 \\
1 & 4 & 1 & 0
\end{bmatrix}
\]

\[
G_{Bs_i} = \begin{bmatrix}
P_{i-3} \\
P_{i-2} \\
P_{i-1} \\
P_i
\end{bmatrix}
\]

\[
b_i(u) \quad b_{i+1}(u) \quad b_{i+2}(u) \quad b_{i+3}(u)
\]
How to Draw Spline Curves

• Basis matrix eqn. allows same code to draw any spline type
• **Method 1: brute force**
  
  – Calculate the coefficients
  
  – For each cubic segment, vary $u$ from 0 to 1 (fixed step size)
  
  – Plug in $u$ value, matrix multiply to compute position on curve
  
  – Draw line segment from last position to current position

\[
\begin{bmatrix}
  x \\
  y \\
  z
\end{bmatrix} = \begin{bmatrix}
  u^3 & u^2 & u & 1
\end{bmatrix}
\begin{bmatrix}
  -s & 2-s & s-2 & s \\
  2s & s-3 & 3-2s & -s \\
  -s & 0 & s & 0 \\
  0 & 1 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
  x_1 & y_1 & z_1 \\
  x_2 & y_2 & z_2 \\
  x_3 & y_3 & z_3 \\
  x_4 & y_4 & z_4
\end{bmatrix}
\]

CR basis  control vector
How to Draw Spline Curves

• What’s wrong with this approach?
  – Draws in even steps of \( u \)
  – Even steps of \( u \) \( \neq \) even steps of \( x \)
  – Line length will vary over the curve
  – Want to bound line length
    » too long: curve looks jagged
    » too short: curve is slow to draw
**Drawing Splines, 2**

- **Method 2: recursive subdivision** - vary step size to draw short lines

  \[
  \text{Subdivide}(u_0, u_1, \text{maxlinelength})
  \]

  \[
  \begin{align*}
  \text{umid} &= (u_0 + u_1)/2 \\
  x_0 &= P(u_0) \\
  x_1 &= P(u_1) \\
  \text{if } |x_1 - x_0| &> \text{maxlinelength} \\
  &\quad \text{Subdivide}(u_0, \text{umid}, \text{maxlinelength}) \\
  &\quad \text{Subdivide}(\text{umid}, u_1, \text{maxlinelength}) \\
  \text{else } &\quad \text{drawline}(x_0, x_1)
  \end{align*}
  \]

- **Variant on Method 2** - subdivide based on curvature
  - replace condition in “if” statement with straightness criterion
  - draws fewer lines in flatter regions of the curve

![Diagram of spline drawing process]
In Summary...

- **Summary:**
  - piecewise cubic is generally sufficient
  - define conditions on the curves and their continuity

- **Things to know:**
  - basic curve properties (what are the conditions, controls, and properties for each spline type)
  - generic matrix formula for uniform cubic splines $x(u) = uB^G$
  - given definition derive a basis matrix
Practice Problems

Write the equation for Catmull-Rom splines in matrix form, assuming that $s=1$. Label the geometry (basis) matrix and the control variables.

How do you guarantee $C^0$ continuity between two adjacent Catmull-Rom splines? $C^1$ continuity? Give an example of control points for two adjacent curves which have $C^0$ and $C^1$ continuity.
Practice Problems

There are a variety of different types of curves mentioned in these slides (Line segments, Hermite splines, Bezier splines, Catmull-Rom splines, Natural Cubic Splines, and B-splines) For each type of curve mentioned in these slides, list pros and cons of using that curve to create animations.