CS229 Handout: Splines Quick Reference

1 Properties

The table below compares some of the cubic splines used in animation. If a curve is a sequence of cubic splines describing how a single parameter (e.g. position along the x-axis) varies with time, then:

- **Continuity** is continuity of a curve at the join points between cubics,
- **Interpolation** indicates whether or not the cubics interpolate their control points.
- **Local** indicates that modifying control point positions has only a local effect; modifying a single control point will affect only a small number of neighboring cubics.
- **Parameters** is the list of control parameters that define cubic i. Parameters P_i for $i = 0, ..n-1$ are positions we would like to interpolate and parameters R_i are velocities at positions P_i. Parameters ϕ_i are positions of the B-Spline control points. Because B-Splines are not interpolating splines, these points will not be identical to positions P_i.

Spline Type	Continuity	Interpolation	Local	Parameters
Natural Cubic Spline	C2	Yes	No	$P_0, ..., P_{n-1}$
Hermite Spline	C1	Yes	Yes	$P_i, P_{i+1}, R_i, R_{i+1}$
Catmull-Rom Spline	C1	Yes	Yes	$P_{i-1}, P_i, P_{i+1}, P_{i+2}$
B-Spline	C2	No	Yes	$\phi_{i-3}, \phi_{i-2}, \phi_{i-1}, \phi_i$

2 Spline Functions

In the sections below, parameter $Y_i(t_i)$ indicates the value of the ith cubic at time t_i. Time parameter t_i has been scaled to run from 0 to 1 for the curve segment covered by cubic i.

2.1 Natural Cubic Spline

Each section of a natural cubic spline is described by a general cubic function of time:

$$Y_i(t_i) = a_i t_i^3 + b_i t_i^2 + c_i t_i + d_i \tag{1}$$

Parameters a_i, b_i, c_i, and d_i must be deterimined by setting up and solving a matrix equation using data from the entire curve. The matrix equation should ensure continuity in positions, velocities, and accelerations at all join points.

2.2 Hermite Splines

Each cubic in a Hermite Spline is fully determined by its local parameters:

$$Y_i(t_i) = (2t_i^3 - 3t_i^2 + 1)P_i + (-2t_i^3 + 3t_i^2)P_{i+1} + (t_i^3 - 2t_i^2 + t_i)R_i + (t_i^3 - t_i^2)R_{i+1} \tag{2}$$

2.3 Catmull-Rom Splines

Catmull-Rom Splines are Hermite Splines where velocities R_i are computed from the data:

$$R_i = \frac{1}{2}(P_{i+1} - P_{i-1}) \tag{3}$$

This amounts to a cubic of the following form, expressed in terms of the data points only:

$$Y_i(t_i) = \frac{1}{2}\left[(-t_i^3 + 2t_i^2 - t_i)P_{i-1} + (3t_i^3 - 5t_i^2 + 2)P_i + (-3t_i^3 + 4t_i^2 + t_i)P_{i+1} + (t_i^3 - t_i^2)P_{i+2}\right] \tag{4}$$

Note that two additional data points must be manufactured to allow computation of the first and last cubic (P_{-1} for cubic 0 and P_n for cubic $n-2$). These points may be chosen, for example, to bring initial and final velocities to zero or to continue the trends of the first few and last few data points.

2.4 B-Splines

B-Splines are fully defined by their local control points:

$$Y_i(t_i) = \frac{1}{6}\left[(-t_i^3 + 3t_i^2 - 3t_i + 1)\phi_i + (3t_i^3 - 6t_i^2 + 4)\phi_{i+1} + (-3t_i^3 + 3t_i^2 + 3t_i + 1)\phi_{i+2} + t_i^3\phi_{i+3}\right] \tag{5}$$

Note, however, that the B-Splines are not interpolating splines. If we insist that the resulting curve should interpolate our data points, the control points ϕ_i must somehow be computed from the data. The process is similar to that for the Natural Cubic Splines. We set up and solve a matrix equation over the entire curve. In the standard $Ax = b$ form of a set of linear equations, b is our vector of data points P_i, x is the vector of control points ϕ_i for which we are solving, and matrix A is constructed using Equation 5.