15-462: Computer Graphics

Math for Computer Graphics

Topics for Today

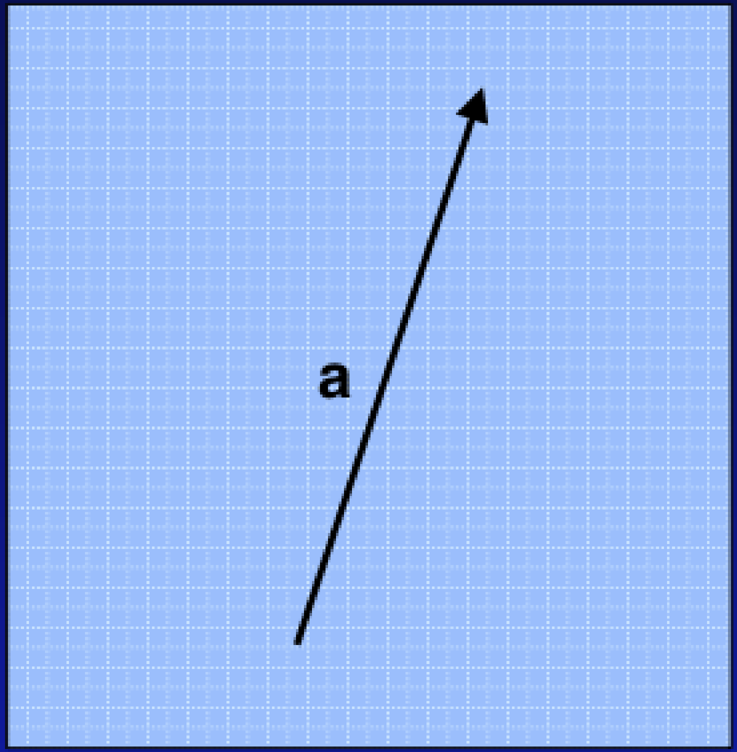
- Vectors
- Equations for curves and surfaces
- Barycentric Coordinates

Topics for Today

- Vectors
  - What is a vector?
  - Coordinate systems
  - Vector arithmetic
  - Dot product
  - Cross product
  - Normal vectors
- Equations for curves and surfaces
- Barycentric Coordinates

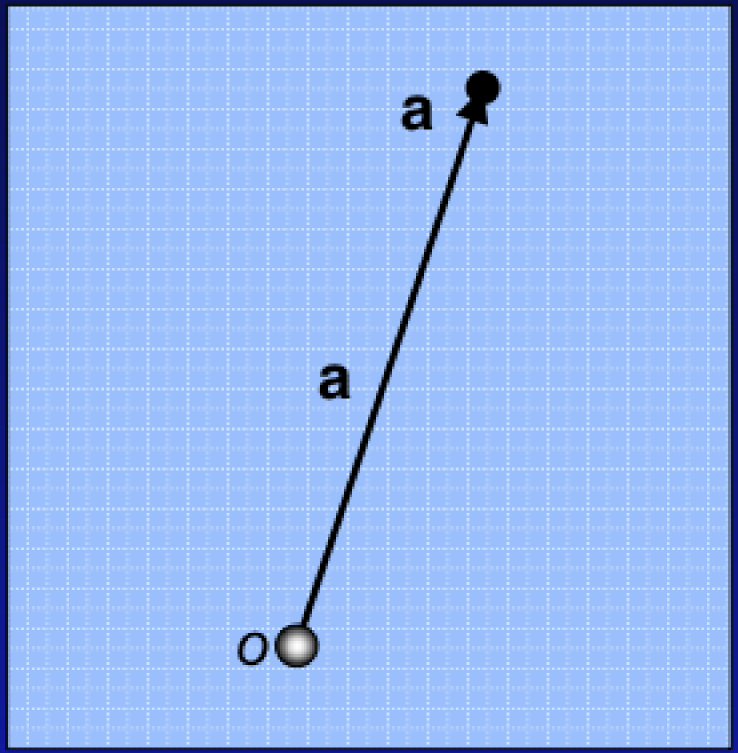
What is a vector?

- A *vector* is a value that describes both a magnitude and a direction. We draw vectors as arrows, and name them with bold letters, e.g. **a**.



What is a vector?

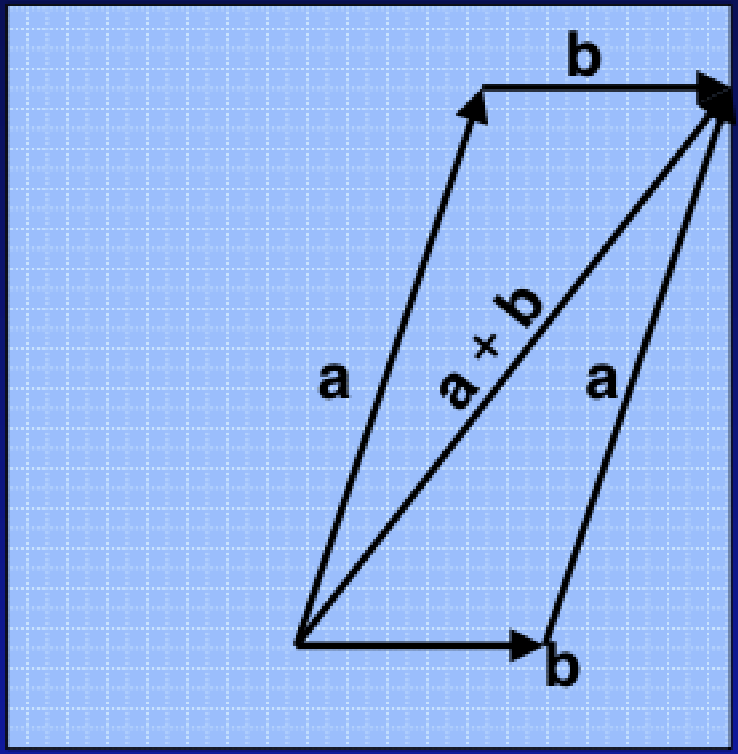
- Vectors themselves contain no information about a starting point.
- We can interpret vectors as *displacements*, instructions to get from one point in space to another.
- We can also interpret vectors as *points*, but in order to do so, we must assume a particular *origin* as the starting point.



Vector arithmetic

- To find the *sum* of two vectors, we place the tail of one to the head of the other. The sum is the vector that completes the triangle.
- Vector addition is commutative:

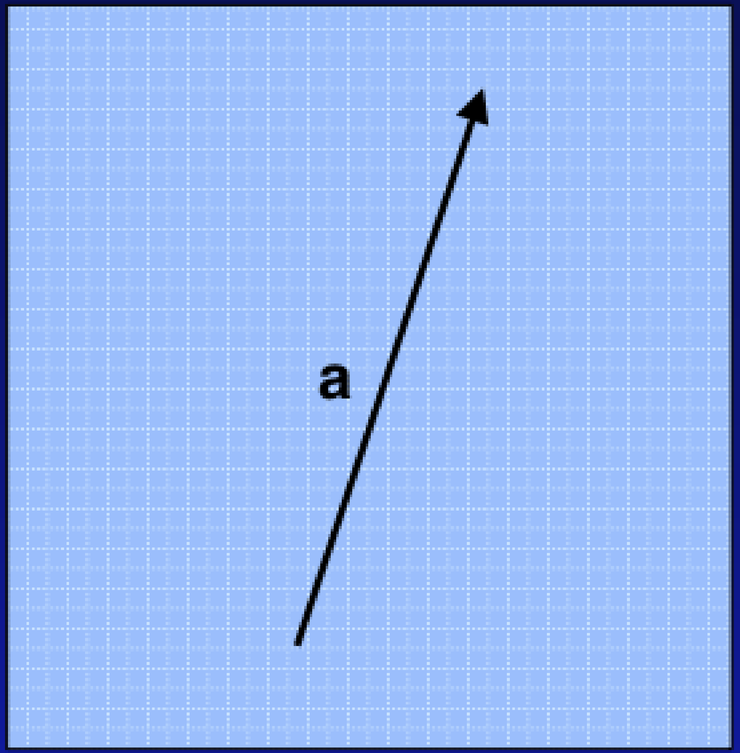
$$\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$$



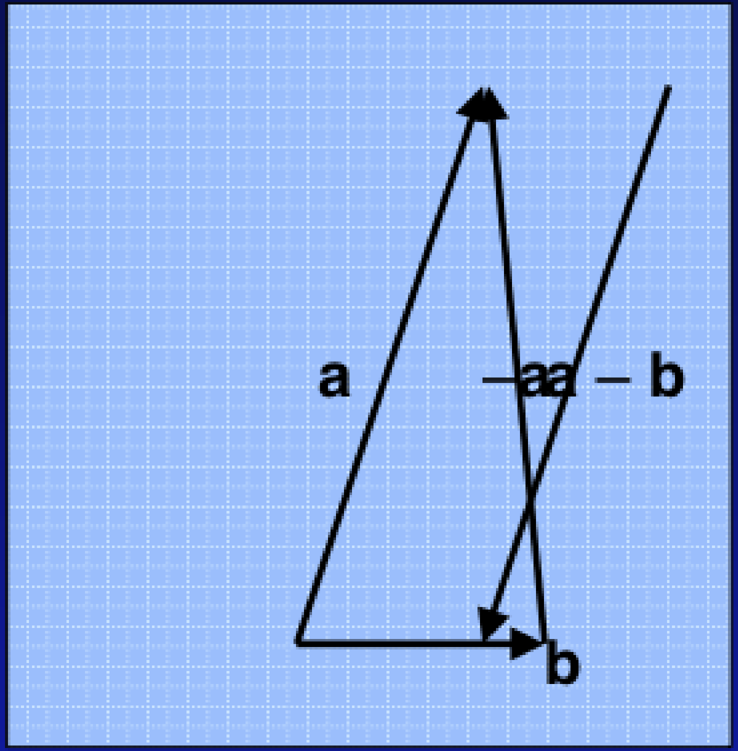
What is a vector?

Some Definitions

- The *magnitude* of vector **a** is the scalar given by ||**a**||.
- A *unit vector* is any vector whose magnitude is one.
- The *zero vector*, **0**, has a magnitude of zero, and its direction is undefined.
- Two vectors are equal if and only if they have equal magnitudes and point in the same direction.



Vector arithmetic



- We define the *unary minus* (negative) such that

$$-\mathbf{a} + \mathbf{a} = \mathbf{0}$$

- We can then define *subtraction* as

$$\mathbf{a} - \mathbf{b} \equiv -\mathbf{b} + \mathbf{a}$$

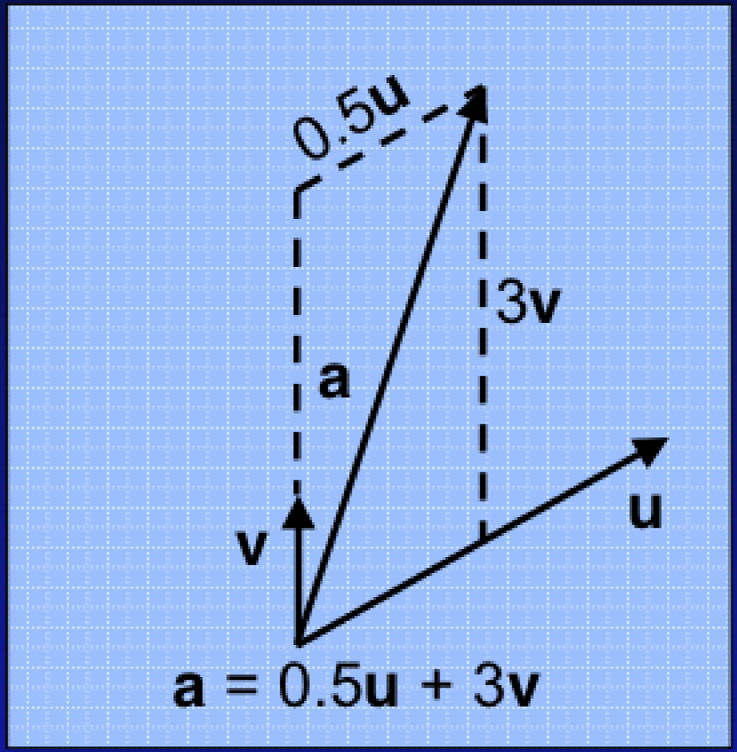
- This gives the vector from the end of **b** to the end of **a** if both have the same origin.

Coordinate systems

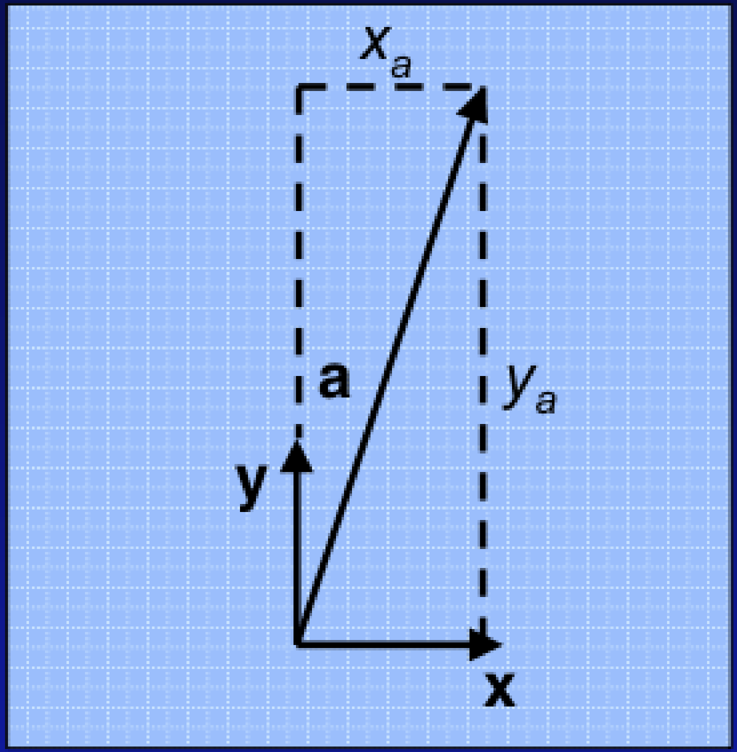
- A vector can be multiplied by a scalar to scale the vector's magnitude without changing its direction:

$$||k\mathbf{a}|| = k||\mathbf{a}||$$

- In 2D, we can represent any vector as a unique *linear combination*, or weighted sum, of any two non-parallel *basis vectors*.
- 3D requires three non-parallel, non-coplanar basis vectors.



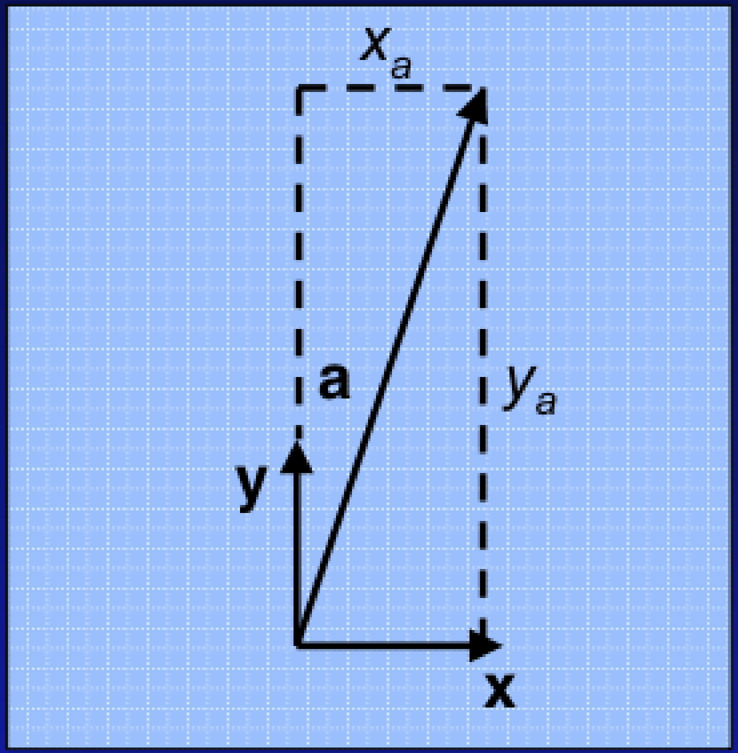
Coordinate systems



- Basis vectors that are unit vectors at right angles to each other are called *orthonormal*.
- The **x**-**y** *Cartesian* coordinate system is a special orthonormal system.
- Vectors are commonly represented in terms of their Cartesian coordinates:

$$\mathbf{a} = \left(x_a, y_a \right) \qquad \mathbf{a} = \begin{bmatrix} x_a \\ y_a \end{bmatrix} \qquad \mathbf{a}^T = \begin{bmatrix} x_a & y_a \end{bmatrix}$$

Coordinate systems



- Vectors expressed by orthonormal coordinates

$$\mathbf{a} = (x_a, y_a)$$

have the very useful property that their magnitudes can by calculated according to the Pythagorean Theorem:

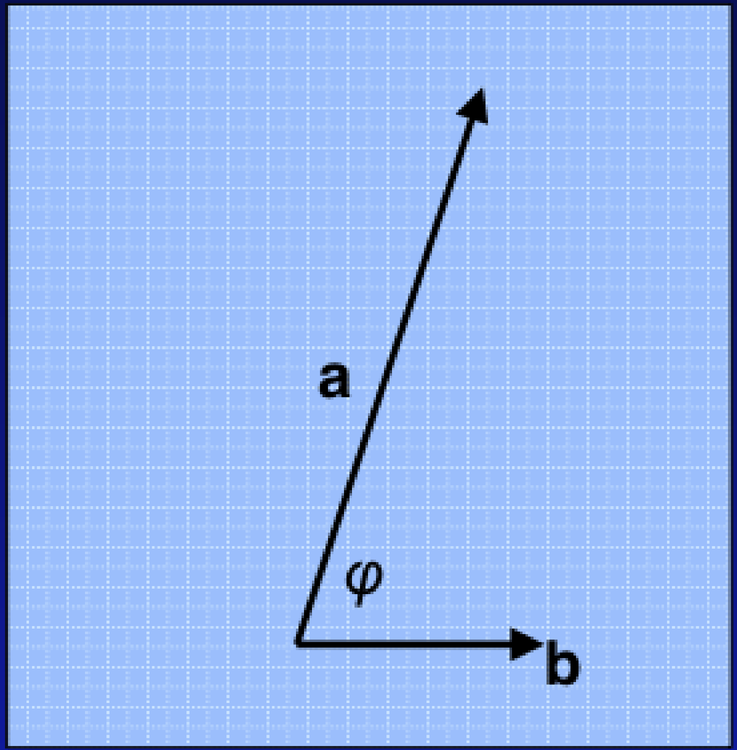
$$||\mathbf{a}|| = \sqrt{x_a^{\ 2} + y_a^{\ 2}}$$

Dot product

- We can multiply two vectors by taking the *dot product*.
- The dot product is defined as

$$\mathbf{a} \cdot \mathbf{b} = ||\mathbf{a}||\ ||\mathbf{b}|| \cos \varphi$$

  where φ is the angle between the two vectors.
- Note that the dot product takes two vectors as arguments, but it is often called the *scalar product* because its result is a scalar.



Dot product

Some cool properties:

- It's often useful in graphics to know the cosine of the angle between two vectors, and we can find it with the dot product:

$$\cos \varphi = \mathbf{a} \cdot \mathbf{b} \,/\, (\|\mathbf{a}\|\,\|\mathbf{b}\|)$$

- We can use the dot product to find the *projection* of one vector onto another. The scalar $\mathbf{a}\rightarrow\mathbf{b}$ is the magnitude of the vector **a** projected at a right angle onto vector **b**, and

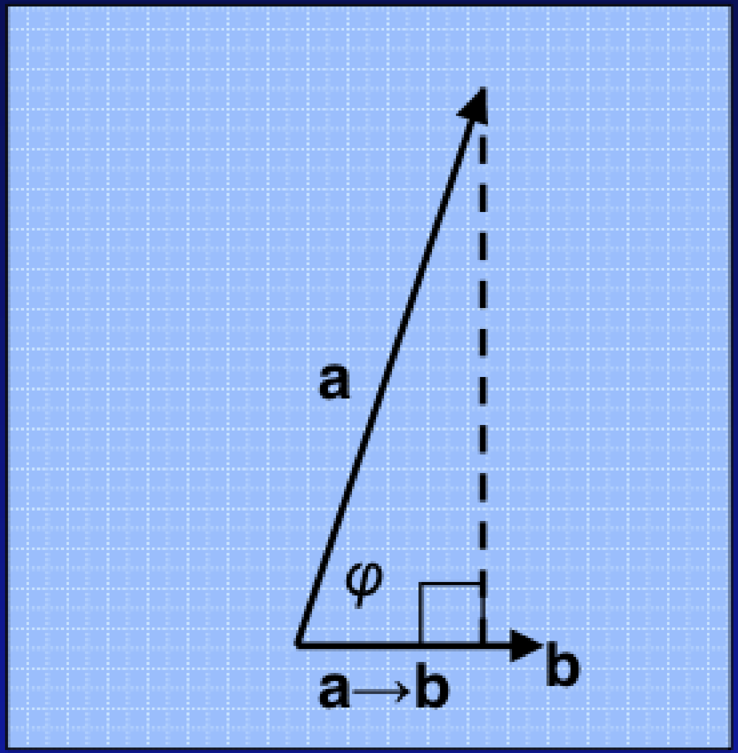
$$\mathbf{a}\rightarrow\mathbf{b} = \|\mathbf{a}\| \cos \varphi = \mathbf{a} \cdot \mathbf{b} \,/\, \|\mathbf{b}\|$$

- Dot products are commutative and distributive:

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$$

$$\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$$

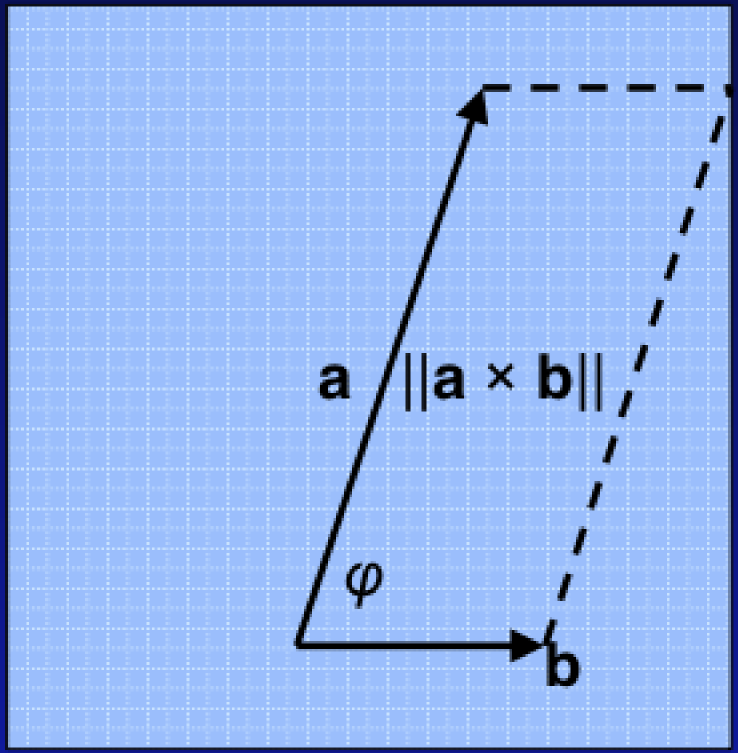
$$(k\mathbf{a}) \cdot \mathbf{b} = \mathbf{a} \cdot (k\mathbf{b}) = k(\mathbf{a} \cdot \mathbf{b})$$



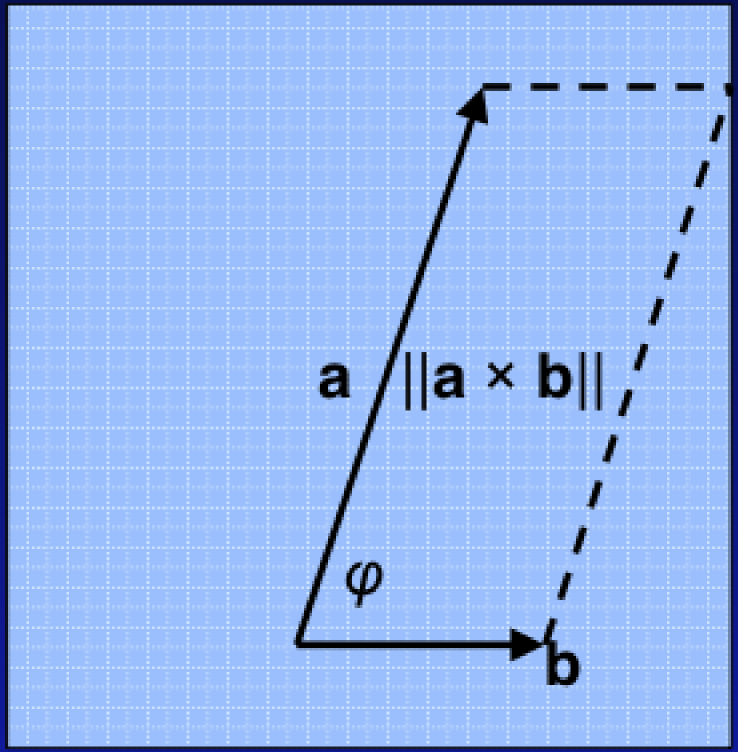
Cross product

- The *cross product* is another vector multiplication operation, usually used only for 3D vectors.
- The direction of $\mathbf{a} \times \mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.
- The magnitude is equal to the area of the parallelogram formed by the two vectors. It is given by

$$||\mathbf{a} \times \mathbf{b}|| = ||\mathbf{a}||\ ||\mathbf{b}|| \sin \varphi$$



Cross product



Some cool properties:

- Cross products are distributive:

$$\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$$

$$(k\mathbf{a}) \times \mathbf{b} = \mathbf{a} \times (k\mathbf{b}) = k(\mathbf{a} \times \mathbf{b})$$

- Cross products are intransitive; in fact,

$$\mathbf{a} \times \mathbf{b} = -\,\mathbf{b} \times \mathbf{a}$$

- Because of the sine in the magnitude calculation, for all **a**,

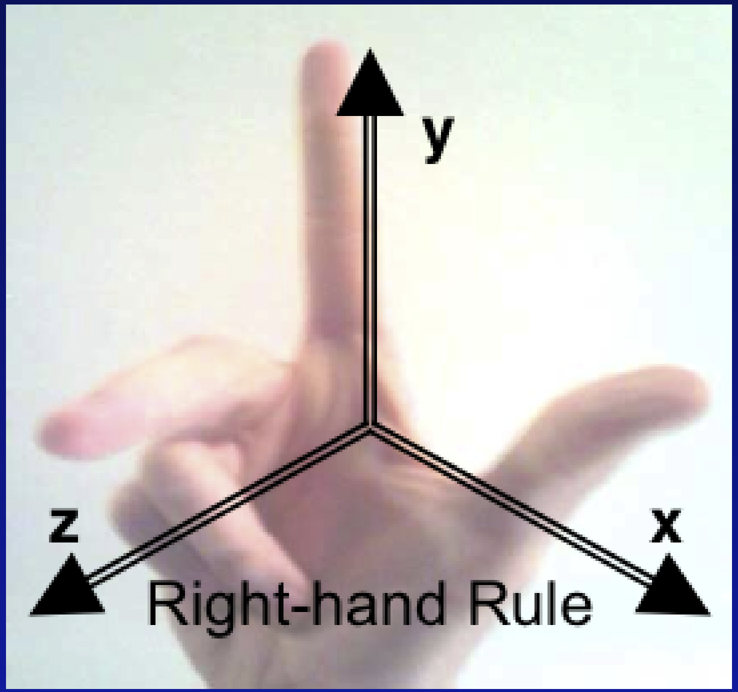
$$\mathbf{a} \times \mathbf{a} = \mathbf{0}$$

- In **x-y-z** Cartesian space,

$$\mathbf{x} \times \mathbf{y} = \mathbf{z} \qquad \mathbf{y} \times \mathbf{z} = \mathbf{x} \qquad \mathbf{z} \times \mathbf{x} = \mathbf{y}$$

Cross product

- As defined on previous slides, the direction of the cross product is ambiguous.
- The *left-hand rule* and the *right-hand rule* distinguish the two choices.
- If **a** points in the direction of your thumb and **b** points in the direction of your index finger, **a** × **b** points in the direction of your middle finger.
- Of the two, the right-hand rule is the predominant convention.



Normal vectors

- A *normal vector* is a vector perpendicular to a surface. A *unit normal* is a normal vector of magnitude one.
- Normal vectors are important to many graphics calculations.
- If the surface is a polygon containing the points **a**, **b**, and **c**, one normal vector

$$\mathbf{n} = (\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{a})$$

- This vector points *into* the polygon if **a**, **b**, and **c** are arranged clockwise; it points outward if they are arranged counterclockwise.

Vectors

Chalkboard examples:

- Cartesian vector addition
- Cartesian dot product
- Cartesian cross product

Topics for Today

- Vectors
- Equations for curves and surfaces
  - Implicit equations
  - Parametric equations
- Barycentric Coordinates

Implicit equations

- *Implicit equations* are a way to define curves and surfaces.
- In 2D, a curve can be defined by

$$f(x,y) = 0$$

  for some scalar function f of x and y.
- In 3D, a surface can be defined by

$$f(x,y,z) = 0$$

  for some scalar function f of x, y, and z.

Implicit equations

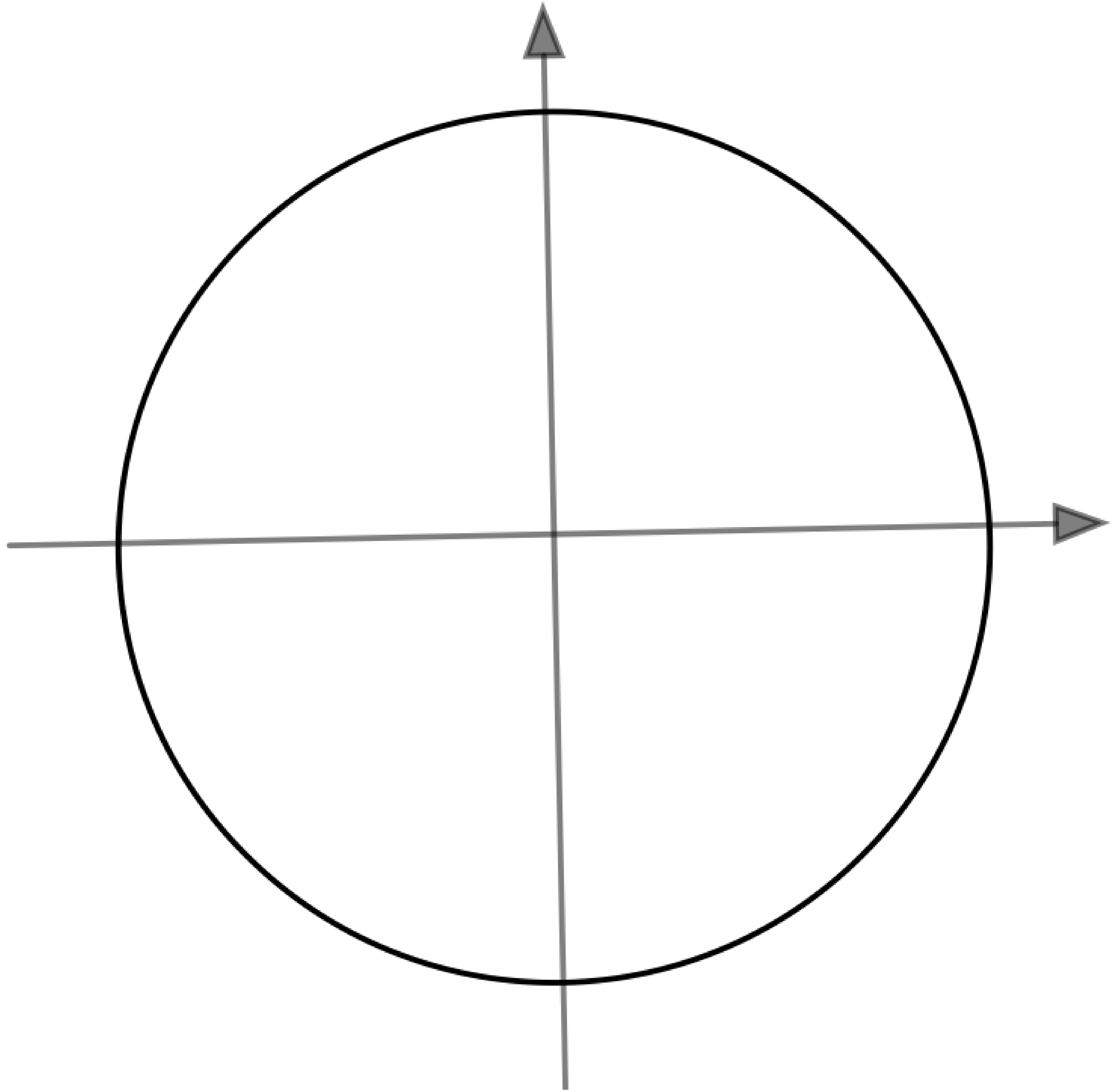
- The function f evaluates to 0 at every point on the curve or surface, and it evaluates to a non-zero real number at all other points.
- Multiplying f by a non-zero coefficient preserves this property, so we can rewrite

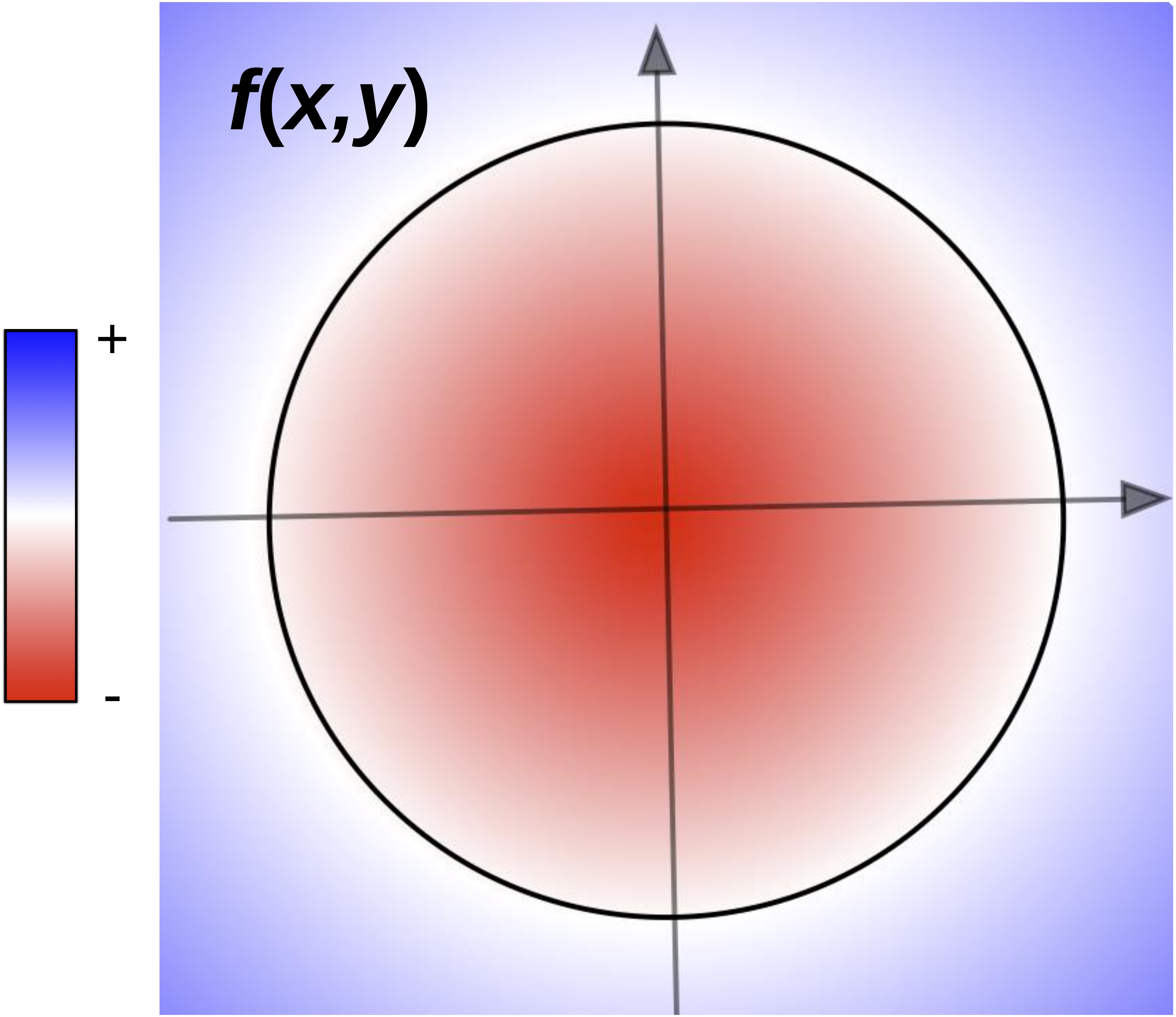
$$f(x,y) = 0$$

as $kf(x,y) = 0$

for any non-zero k.

- The implied curve is unaffected.





f(x,y)
+
-

Implicit equations

Chalkboard examples:

- Implicit 2D circle
- Implicit 2D line
- Implicit 3D plane

Implicit equations

- We call these equations “implicit” because although they imply a curve or surface, they cannot explicitly generate the points that comprise it.
- In order to generate points, we need another form…

Parametric equations

- *Parametric equations* offer the capability to generate continuous curves and surfaces.
- For curves, parametric equations take the form

$$x = f(t) \qquad y = g(t) \qquad z = h(t)$$

- For 3D surfaces, we have

$$x = f(s,t) \quad y = g(s,t) \quad z = h(s,t)$$

Parametric equations

- The *parameters* for these equations are scalars that range over a continuous (possibly infinite) interval.
- Varying the parameters over their entire intervals smoothly generates every point on the curve or surface.

Implicit equations

Chalkboard examples:

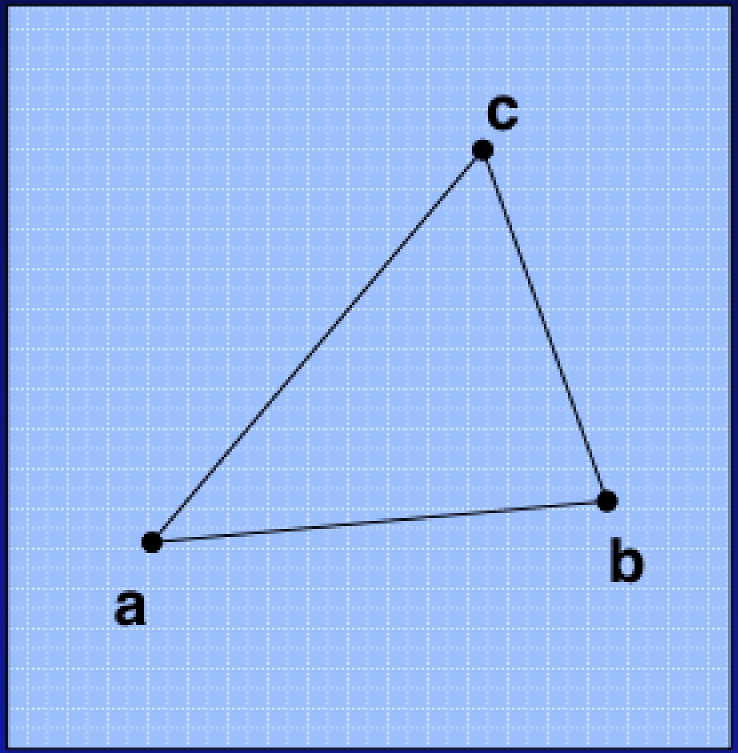
- Parametric 3D line
- Parametric sphere

Topics for Today

- Vectors
- Equations for curves and surfaces
- Barycentric Coordinates
  - Why barycentric coordinates?
  - What are barycentric coordinates?

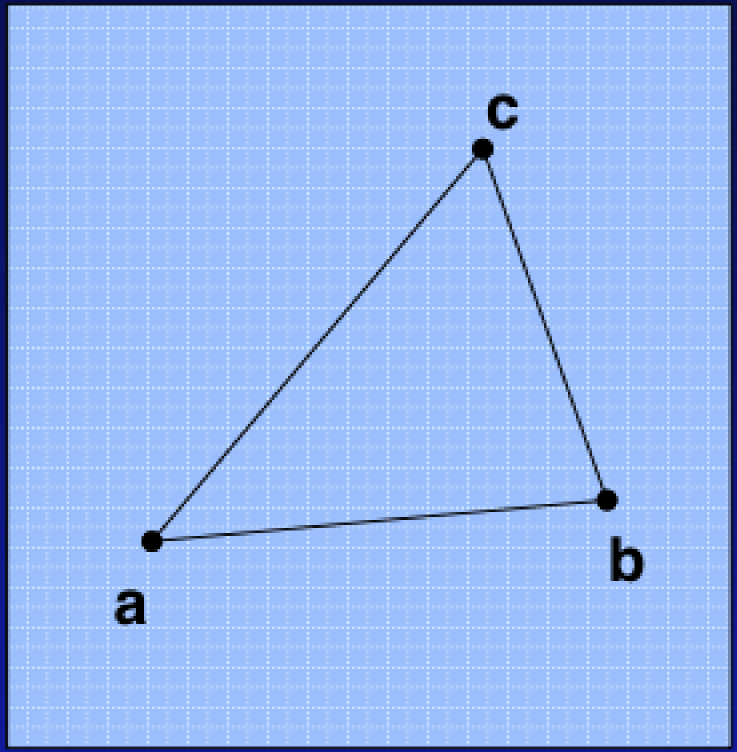
Why barycentric coordinates?

- Triangles are the fundamental primitive used in 3D modeling programs.
- Triangles are stored as a sequence of three vectors, each defining a vertex.
- Often, we know information about the vertices, such as color, that we'd like to interpolate over the whole triangle.



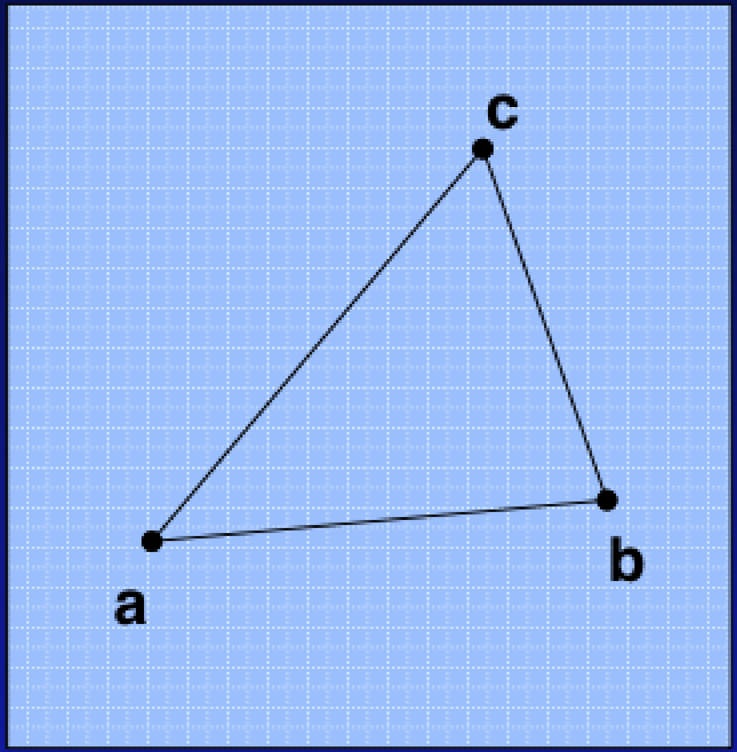
What are barycentric coordinates?

- The simplest way to do this interpolation is *barycentric coordinates*.
- The name comes from the Greek word *barus* (heavy) because the coordinates are weights assigned to the vertices.
- Point **a** on the triangle is the origin of the non-orthogonal coordinate system.
- The vectors from **a** to **b** and from **a** to **c** are taken as basis vectors.

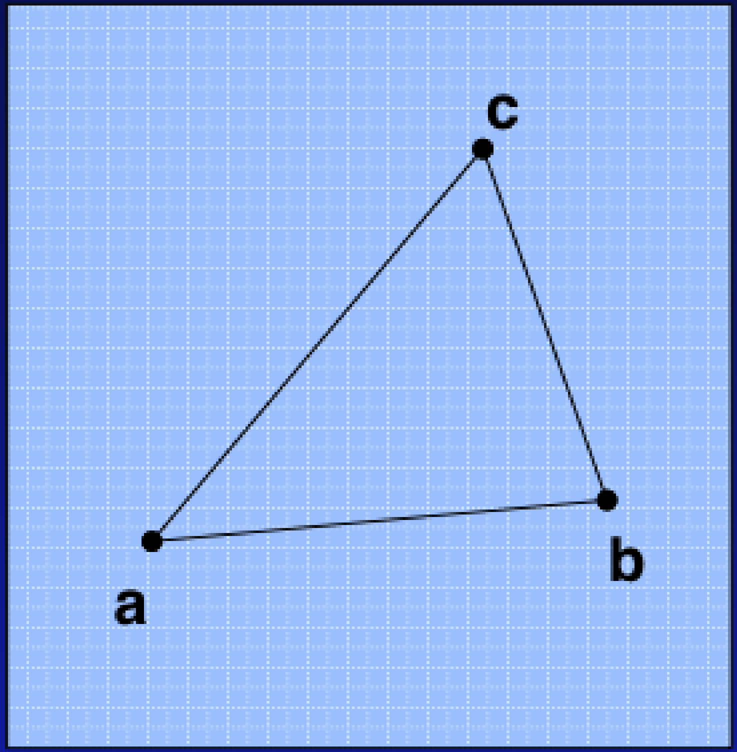


What are barycentric coordinates?

- We can express any point **p** coplanar to the triangle as:
  $\mathbf{p} = \mathbf{a} + \beta(\mathbf{b} - \mathbf{a}) + \gamma(\mathbf{c} - \mathbf{a})$
- Typically, we rewrite this as:
  $\mathbf{p}(\alpha,\beta,\gamma) = \alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c}$
  where $\alpha \equiv 1 - \beta - \gamma$
- $\mathbf{a} = \mathbf{p}(1,0,0)$, $\mathbf{b} = \mathbf{p}(0,1,0)$, $\mathbf{c} = \mathbf{p}(0,0,1)$



What are barycentric coordinates?



Some cool properties:

- Point **p** is inside the triangle if and only if

$$0 < \alpha < 1,$$
$$0 < \beta < 1,$$
$$0 < \gamma < 1$$

- If one component is zero, **p** is on an edge.
- If two components are zero, **p** is on a vertex.
- The coordinates can be used as weighting factors for properties of the vertices, like color.

Barycentric coordinates

Chalkboard examples:

- Conversion from 2D Cartesian
- Conversion from 3D Cartesian

Practice Problems

You are given point **a** = $[2\ 3]^T$ and point **b** = $[1\ 5]^T$

Write an implicit equation for the line passing through **a** and **b**. Try to do this without referring back to your notes or the slides (derive it from scratch as you would in an exam).

Write a parametric equation for this same line. Again, try not to refer to your notes until you have to.

Find a normal for this line. (It does not have to be a unit magnitude normal.)

Practice Problems

Find an intersection between the line ab on the previous slide and a circle of radius r centered at point c. Plug in some numbers for r and c to check your answer. Hint: you may wish to use the implicit equation for a circle and the parametric equation for the line. Plug your parametric line equation for line ab into the implicit circle equation and solve for the free parameter. You will use the 3D version of this technique in your ray tracer.

Draw a triangle abc. Where do you think a point with barycentric coordinates [0.1 0.3 0.6] would fall?
Barycentric coordinates [0.5 0.0 0.5]?
Barycentric coordinates [-0.3 0.5 0.8]?