# Parametric Curves

**Modeling:**

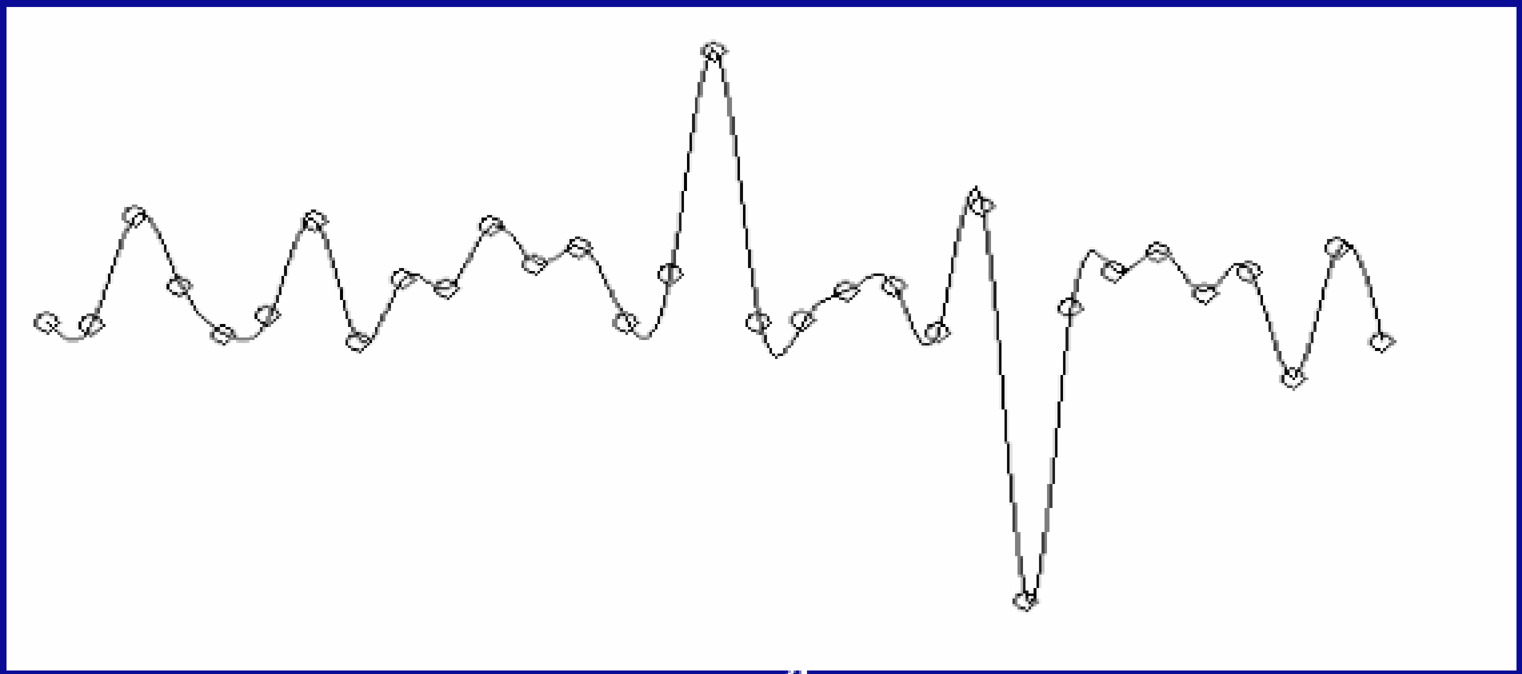
- parametric curves (Splines)
- polygonal meshes

# Modeling Complex Shapes

- We want to build models of very complicated objects
- An equation for a sphere is possible, but how about an equation for a telephone, or a face?
- Complexity is achieved using simple pieces
  - polygons, parametric curves and surfaces, or implicit curves and surfaces
  - This lecture: parametric curves

# What Do We Need From Curves in Computer Graphics?

- Local control of shape (so that easy to build and modify)
- Stability
- Smoothness and continuity
- Ability to evaluate derivatives
- Ease of rendering

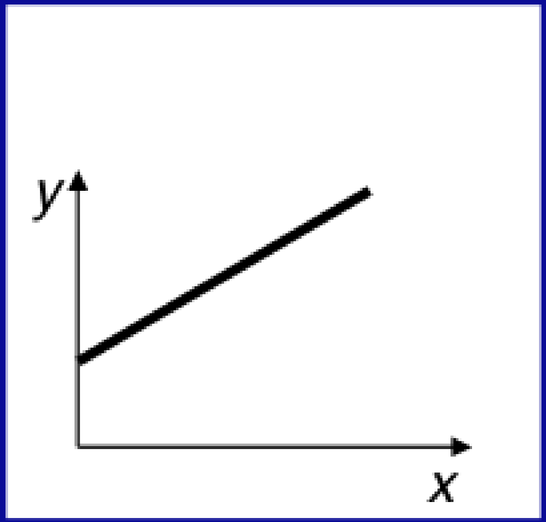


# Curve Representations

- **Explicit: y = f(x)**

$$y = mx + b$$

  - **Easy to generate points**
  - **Must be a function: big limitation—vertical lines?**



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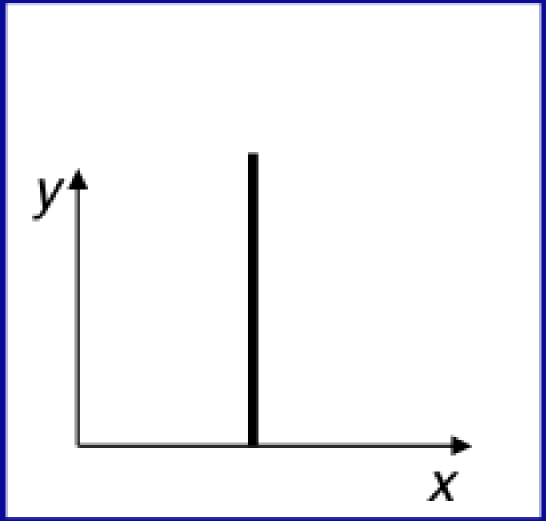
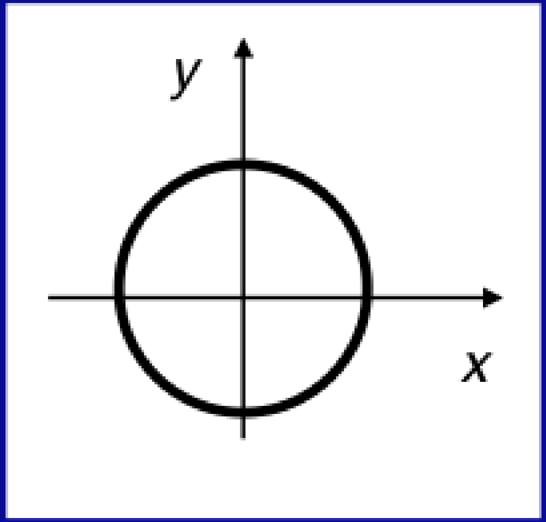



- **Implicit: f(x,y) = 0**

$$x^2 + y^2 - r^2 = 0$$

  +Easy to test if on the curve
  –Hard to generate points



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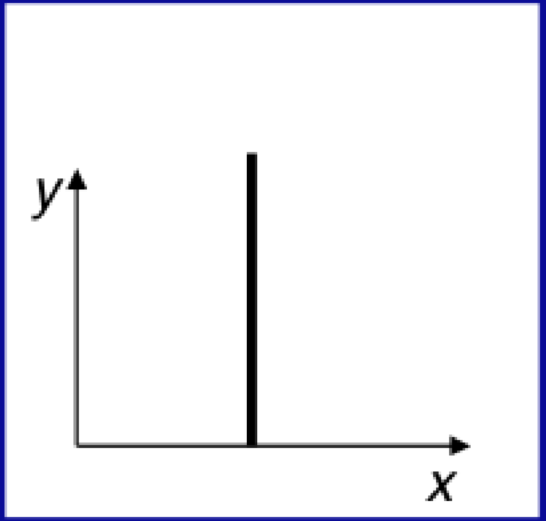


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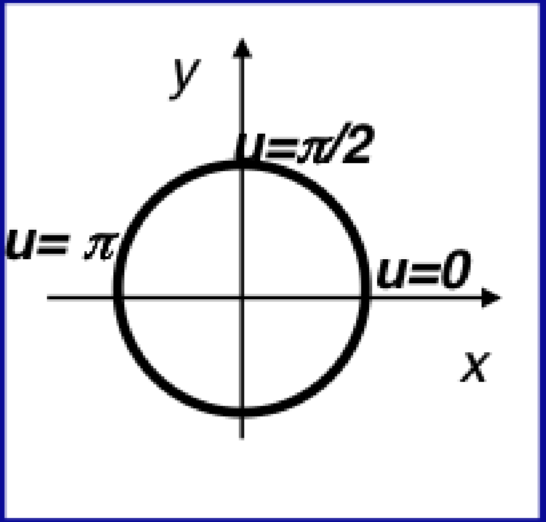
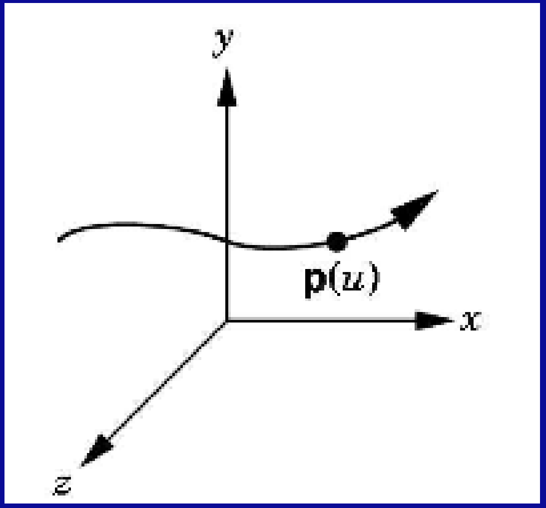



- **Parametric: (x,y) = ( f(u), g(u))**

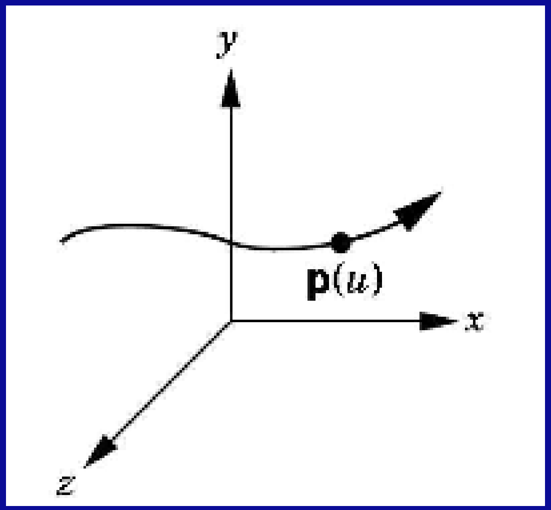
$$(x, y) = (\cos u, \sin u)$$

+Easy to generate points



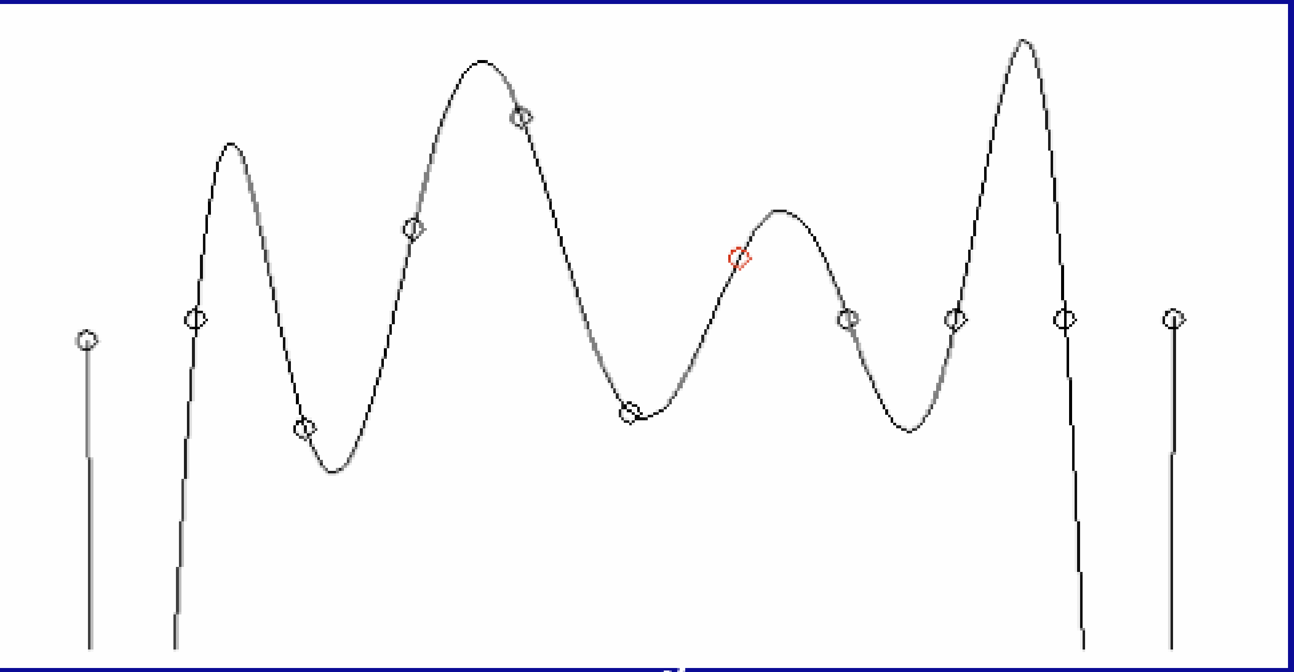
# Parameterization of a Curve

- *Parameterization* of a curve: how a change in u moves you along a given curve in xyz space.



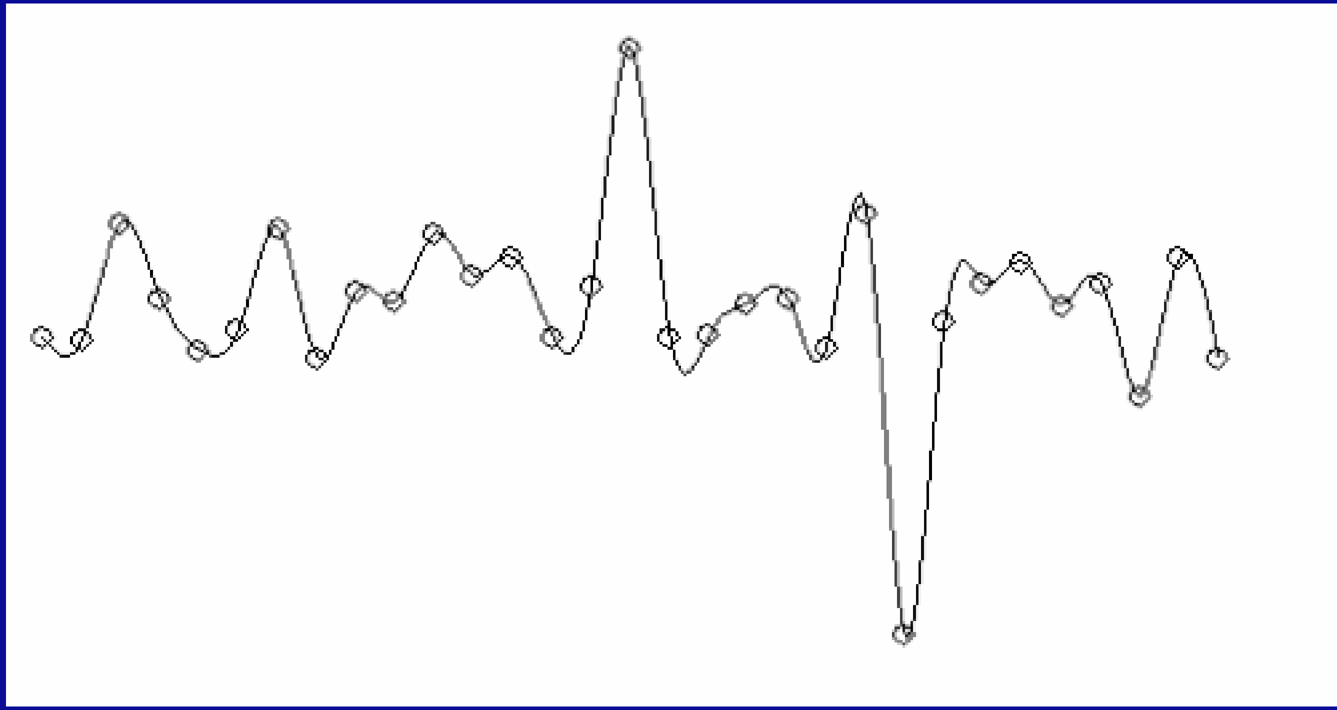
# Polynomial Interpolation

- An *n*-th degree polynomial fits a curve to n+1 points
  - called Lagrange Interpolation
  - result is a curve that is too wiggly, change to any control point affects entire curve (nonlocal) – *this method is poor*
- We usually want the curve to be as smooth as possible
  - minimize the wiggles
  - high-degree polynomials are bad



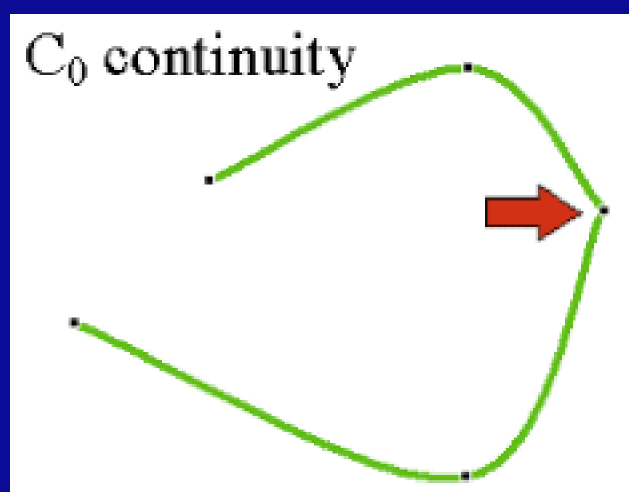
# Splines: Piecewise Polynomials

- A spline is a *piecewise polynomial* - many low degree polynomials are used to interpolate (pass through) the control points
- *Cubic piecewise* polynomials are the most common:
  - piecewise definition gives local control

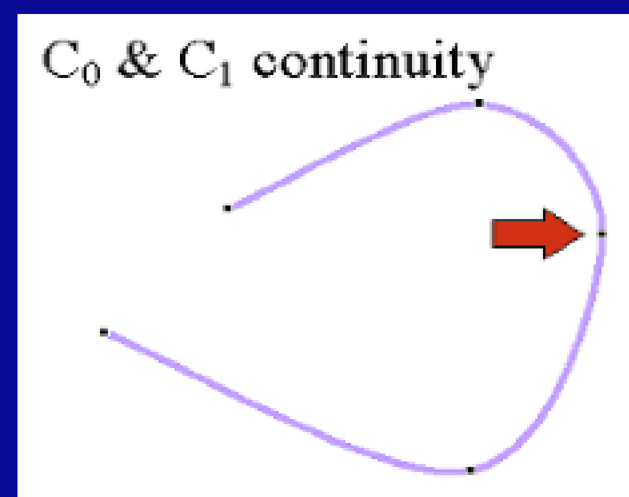


# Piecewise Polynomials

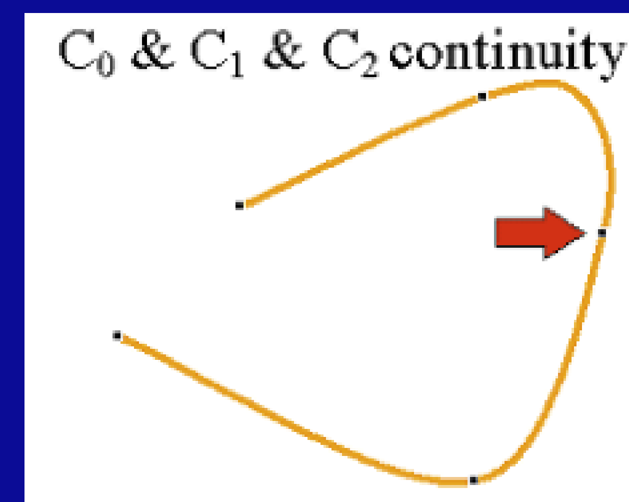
- Spline: lots of little polynomials pieced together
- Want to make sure they fit together nicely




Continuous in position




Continuous in position and tangent vector



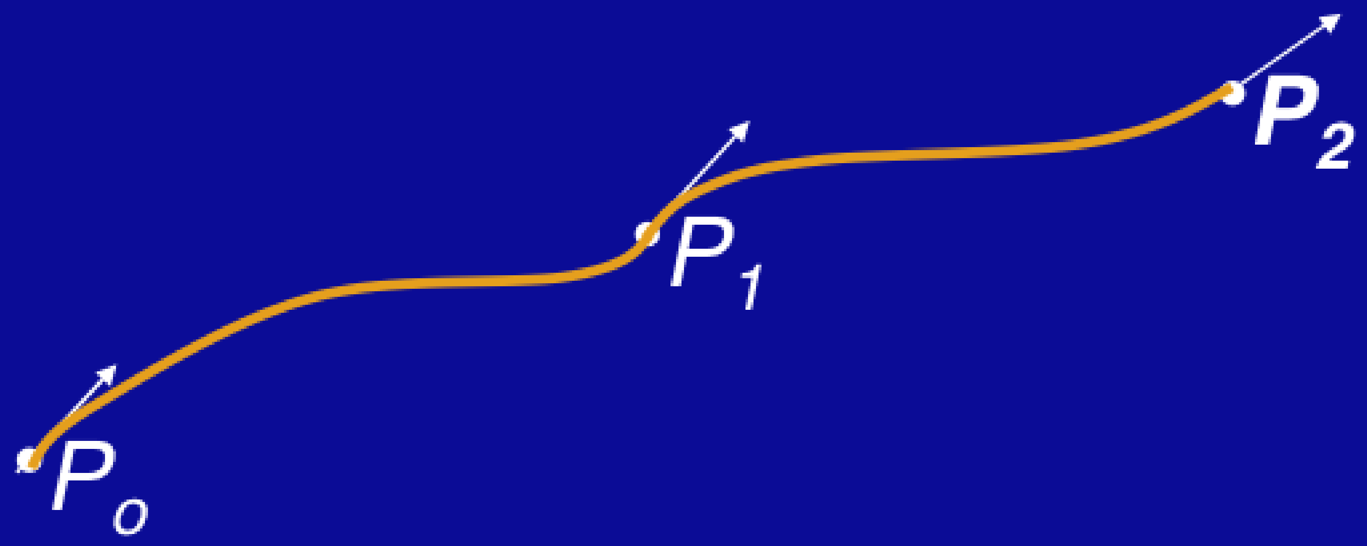

Continuous in position, tangent, and curvature

# Splines

- Types of splines:
  - Hermite Splines
  - Catmull-Rom Splines
  - Bezier Splines
  - Natural Cubic Splines
  - B-Splines
  - NURBS

# Hermite Curves

- Cubic Hermite Splines




**That is, we want a way to specify the end points and the slope at the end points!**

# Splines

chalkboard

# The Cubic Hermite Spline Equation

- Using some algebra, we obtain:

$$p(u) = \begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ \nabla p_1 \\ \nabla p_2 \end{bmatrix}$$

**point that gets drawn**

**basis**

**control matrix (what the user gets to pick)**

- This form typical for splines
  - basis matrix and meaning of control matrix change with the spline type

# The Cubic Hermite Spline Equation

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**point that gets drawn** **basis** **control matrix (what the user gets to pick)**

$$p(u) = \begin{bmatrix} 2u^3 - 3u^2 + 1 \\ -2u^3 + 3u^2 \\ u^3 - 2u^2 + u \\ u^3 - u^2 \end{bmatrix}^T \begin{bmatrix} p_1 \\ p_2 \\ \nabla p_1 \\ \nabla p_2 \end{bmatrix}$$

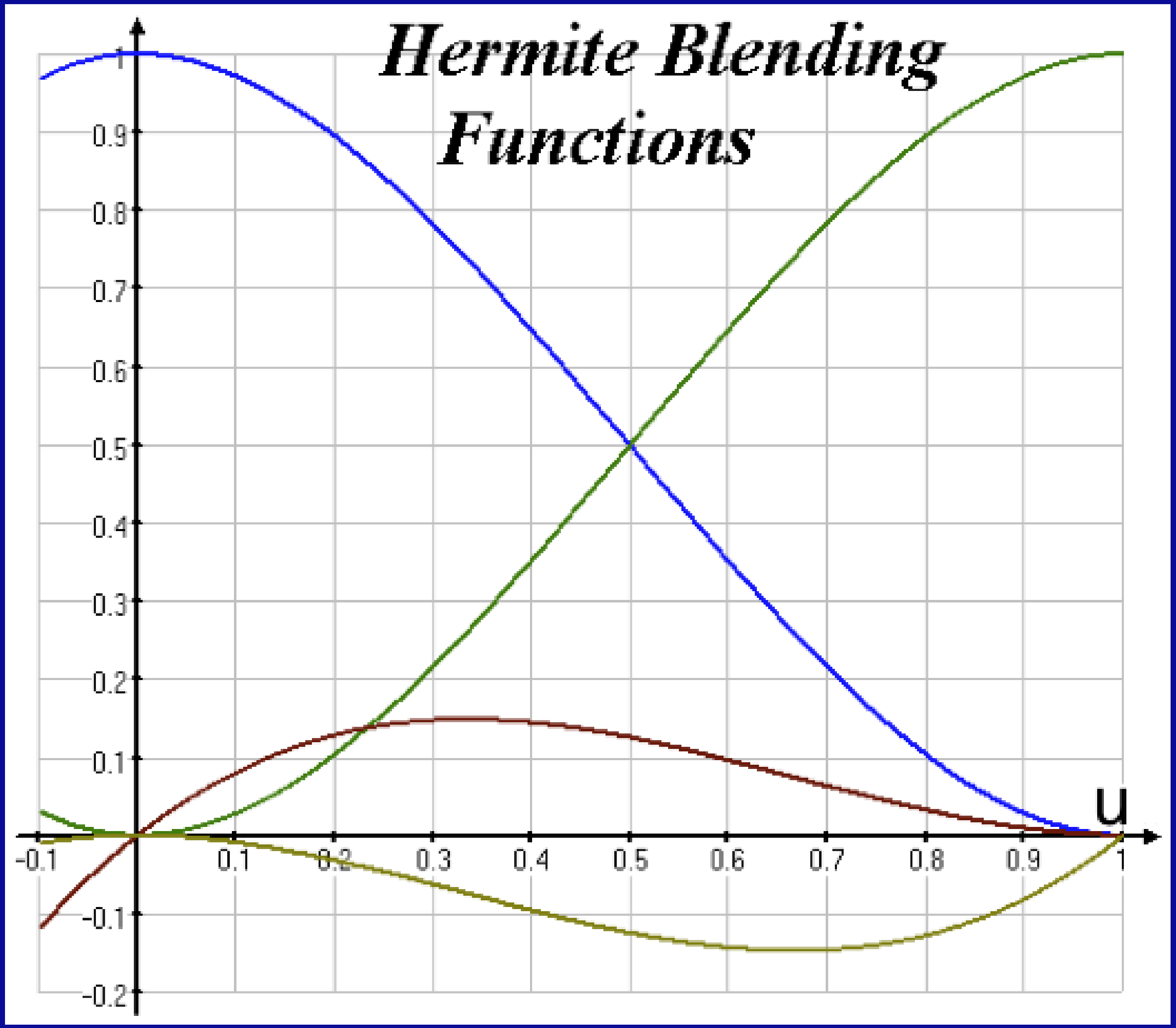
**4 Basis Functions**

# Four Basis Functions for Hermite splines

$$p(u) = \begin{bmatrix} 2u^3 - 3u^2 + 1 \\ -2u^3 + 3u^2 \\ u^3 - 2u^2 + u \\ u^3 - u^2 \end{bmatrix}^T \begin{bmatrix} p_1 \\ p_2 \\ \nabla p_1 \\ \nabla p_2 \end{bmatrix}$$

↑

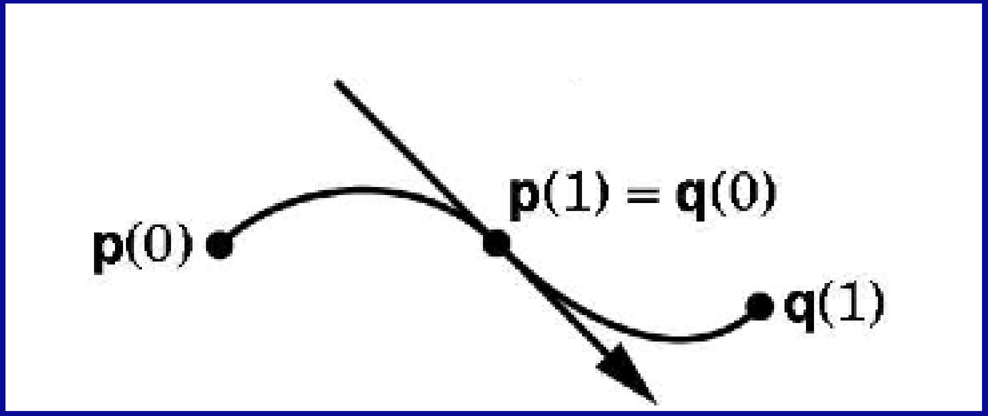
**4 Basis Functions**




**Every cubic Hermite spline is a linear combination (blend) of these 4 functions**

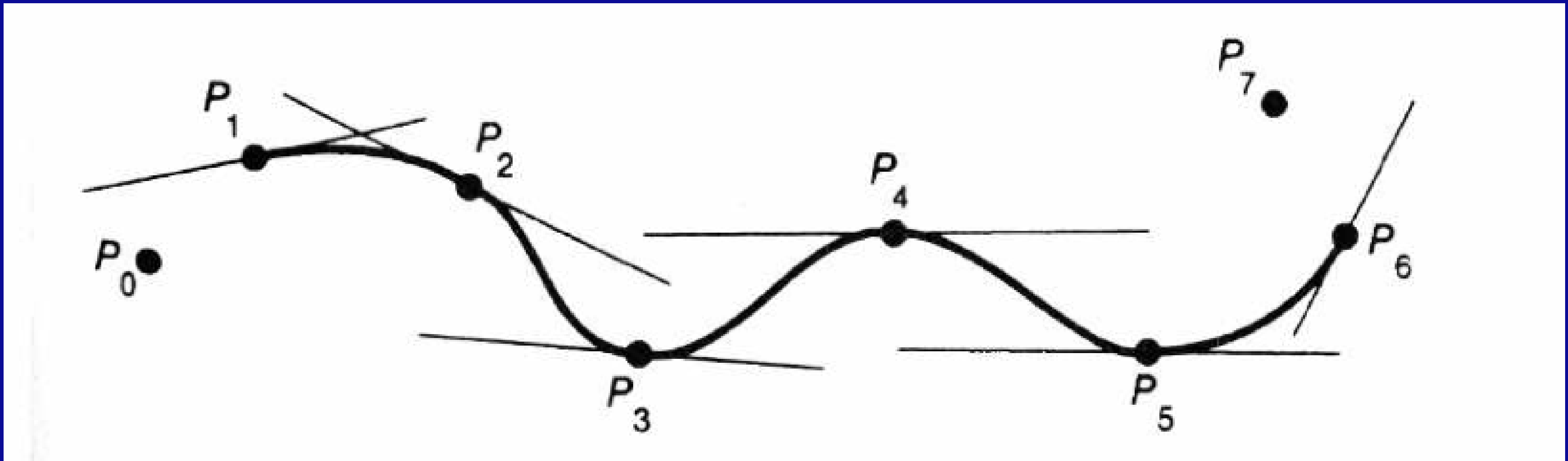
# Piecing together Hermite Curves

- It's easy to make a multi-segment Hermite spline
  - each piece is specified by a cubic Hermite curve
  - just specify the position and tangent at each "joint"
  - the pieces fit together with matched positions and first derivatives
  - gives C1 continuity
- The points that the curve has to pass through are called *knots* or *knot points*



# Catmull-Rom Splines

- With Hermite splines, the designer must specify all the tangent vectors
- Catmull-Rom: an interpolating cubic spline with *built-in $C^1$ continuity*.



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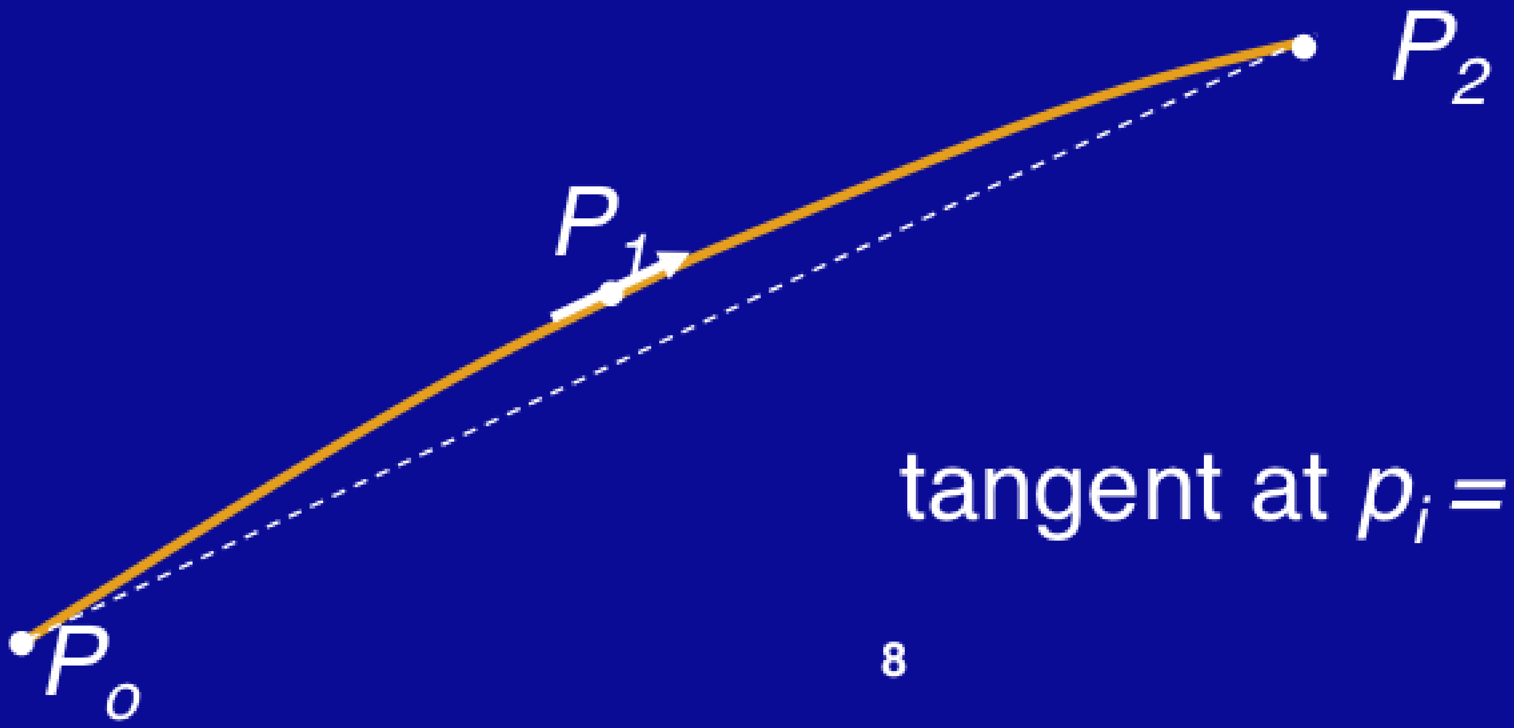


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tangent at $p_i = s(p_{i+1} - p_{i-1})$

# Catmull-Rom Spline Matrix

$$p(u) = \begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix} \begin{bmatrix} -s & 2-s & s-2 & s \\ 2s & s-3 & 3-2s & -s \\ -s & 0 & s & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix}$$

**spline coefficients** **CR basis** **control vector**

- **Derived similarly to Hermite**
- **Parameter s is typically set to s=1/2.**

# Cubic Curves in 3D

- Three cubic polynomials, one for each coordinate
  - $x(u) = a_x u^3 + b_x u^2 + c_x u + d_x$
  - $y(u) = a_y u^3 + b_y u^2 + c_y u + d_y$
  - $z(u) = a_z u^3 + b_z u^2 + c_z u + d_z$
- In matrix notation

$$\begin{bmatrix} x(u) & y(u) & z(u) \end{bmatrix} = \begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix} \begin{bmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \\ d_x & d_y & d_z \end{bmatrix}$$

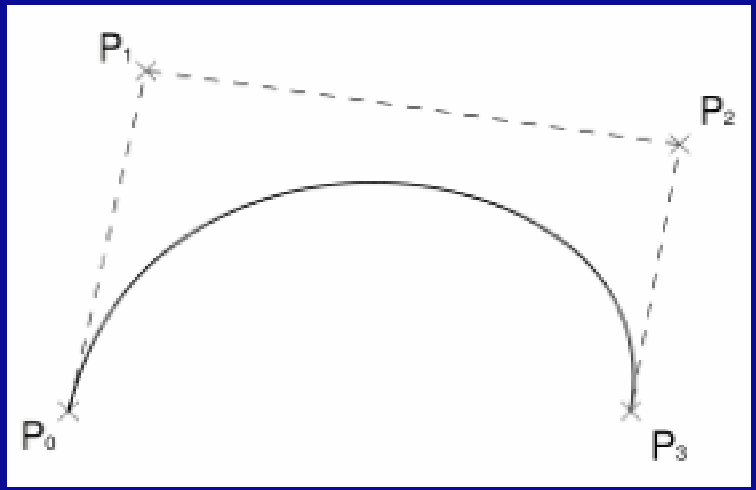
# Catmull-Rom Spline Matrix in 3D

$$\begin{bmatrix} x(u) & y(u) & z(u) \end{bmatrix} = \begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix} \begin{bmatrix} -s & 2-s & s-2 & s \\ 2s & s-3 & 3-2s & -s \\ -s & 0 & s & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \\ x_4 & y_4 & z_4 \end{bmatrix}$$

**CR basis** **control vector**

# Bezier Curves*

- Another variant of the same game
- Instead of endpoints and tangents, four control points
  - points P0 and P3 are on the curve: P(u=0) = P0, P(u=1) = P3
  - points P1 and P2 are off the curve
  - P'(u=0) = 3(P1-P0), P'(u=1) = 3(P3 – P2)
- Convex Hull property
  - curve contained within convex hull of control points
- Gives more control knobs (series of points) than Hermite
- Scale factor (3) is chosen to make "velocity" approximately constant



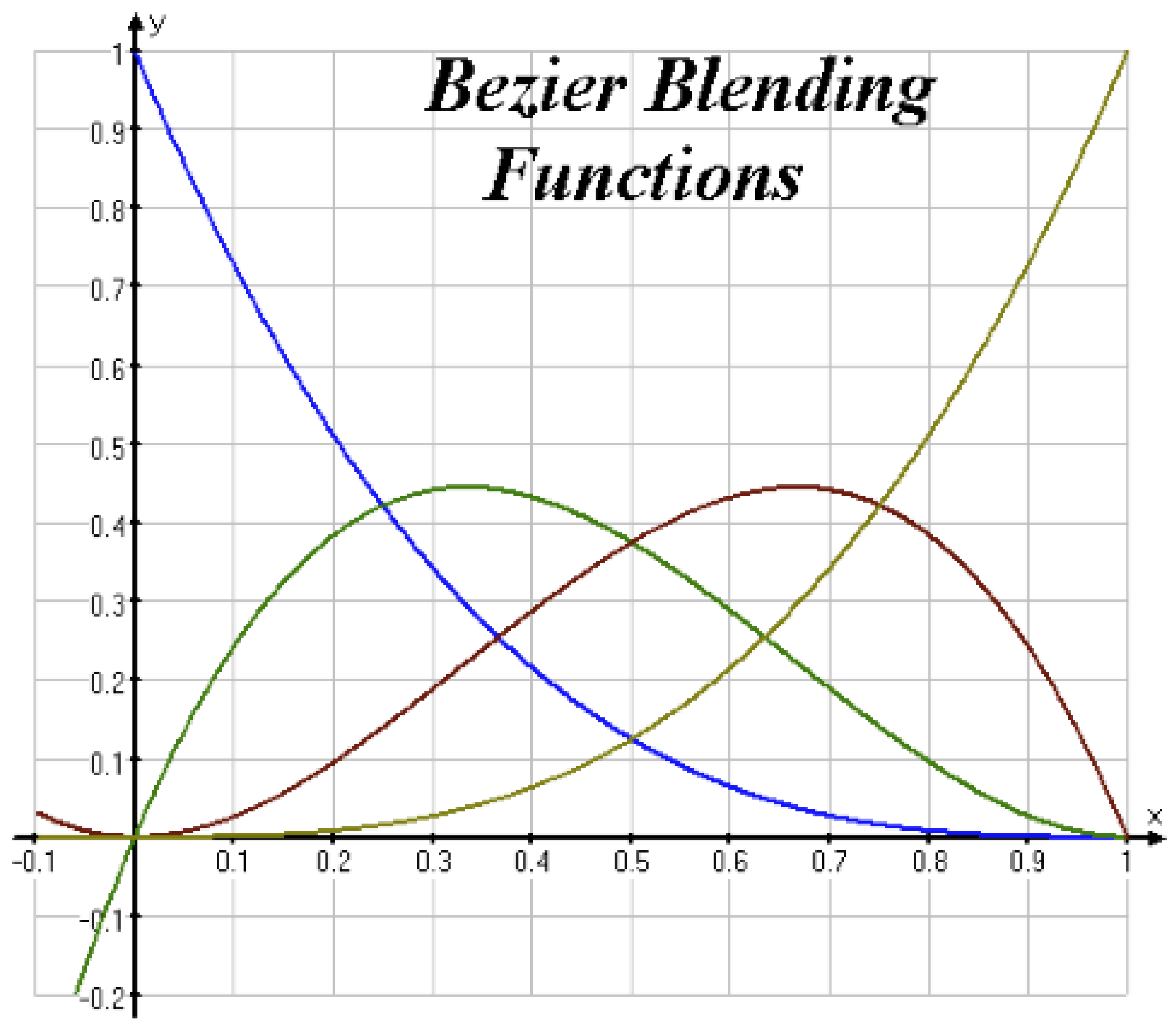
# The Bezier Spline Matrix*

$$[x \;\; y \;\; z] = [u^3 \;\; u^2 \;\; u \;\; 1] \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \\ x_4 & y_4 & z_4 \end{bmatrix}$$

**Bezier basis**

**Bezier control vector**

# Bezier Blending Functions*




$$p(t) = \begin{bmatrix} (1-t)^3 \\ 3t(1-t)^2 \\ 3t^2(1-t) \\ t^3 \end{bmatrix}^T \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix}$$

Also known as the order 4, degree 3 Bernstein polynomials

Nonnegative, sum to 1

The entire curve lies inside the polyhedron bounded by the control points

# Splines with More Continuity?

- How could we get $C^2$ continuity at control points?

- Possible answers:
  - Use higher degree polynomials

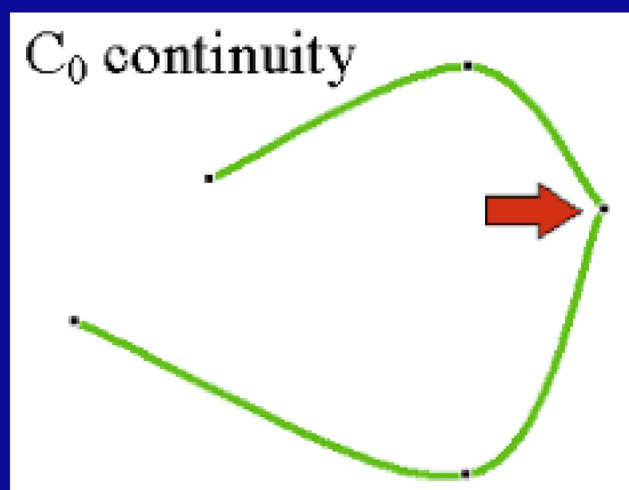
    degree 4 = quartic, degree 5 = quintic, … but these get computationally expensive, and sometimes wiggly
  - Give up local control     natural cubic splines

    A change to any control point affects the entire curve
  - Give up interpolation     cubic B-splines

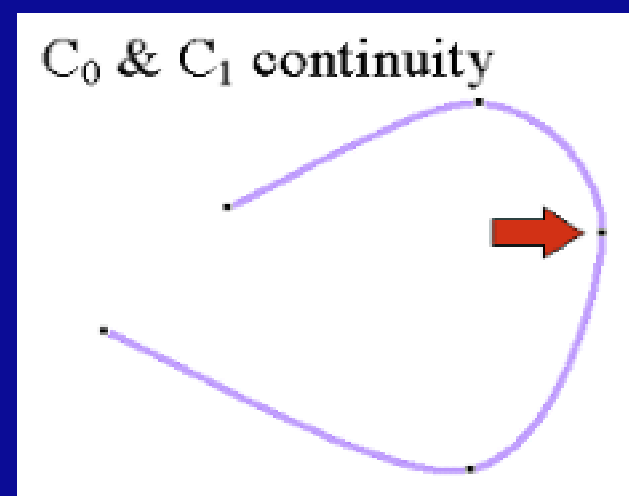
    Curve goes near, but not through, the control points

# Piecewise Polynomials

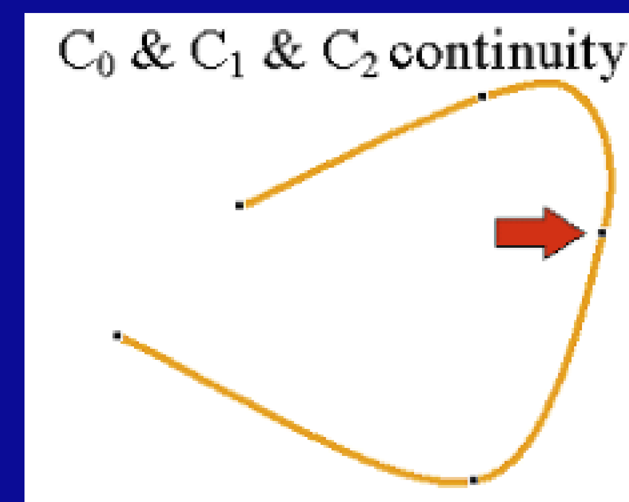
- Spline: lots of little polynomials pieced together
- Want to make sure they fit together nicely




**Continuous in position**




**Continuous in position and tangent vector**




**Continuous in position, tangent, and curvature**

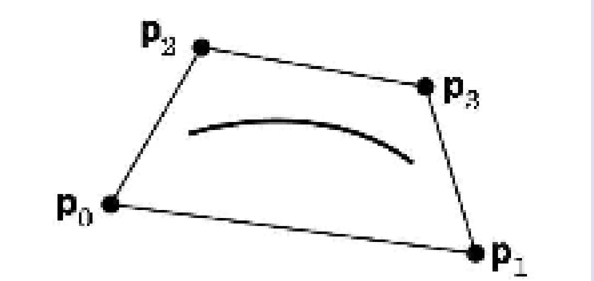
# Comparison of Basic Cubic Splines

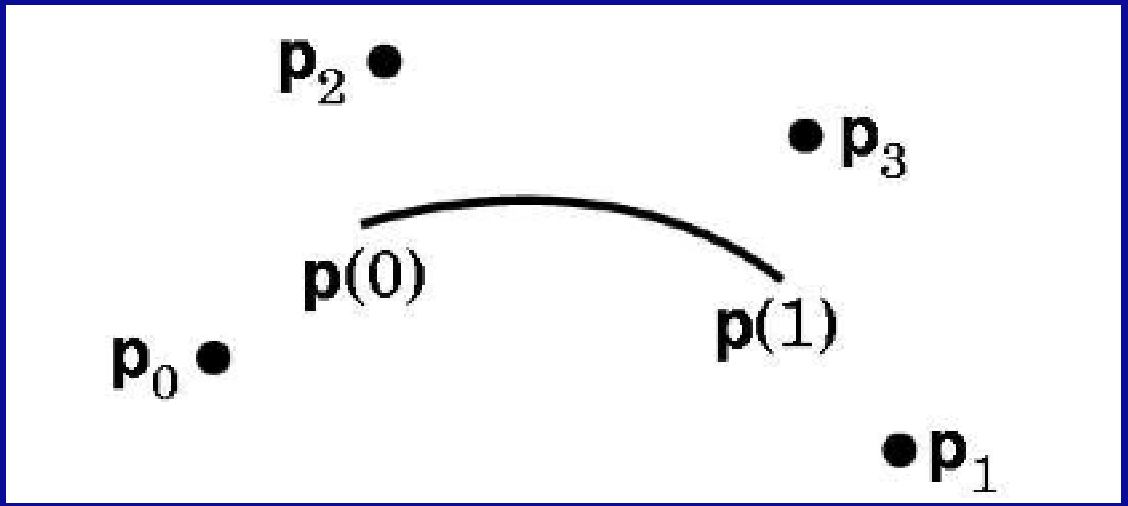
| Type | Local Control | Continuity | Interpolation |
|---|---|---|---|
| Hermite | YES | C1 | YES |
| Bezier | YES | C1 | YES |
| Catmull-Rom | YES | C1 | YES |
| Natural | NO | C2 | YES |
| B-Splines | YES | C2 | NO |

- Summary
  - Can’t get C2, interpolation and local control with cubics

# B-Splines*

- Give up interpolation
  - the curve passes *near* the control points
  - best generated with interactive placement (because it's hard to guess where the curve will go)
- Curve obeys the convex hull property
- C2 continuity and local control are good compensation for loss of interpolation



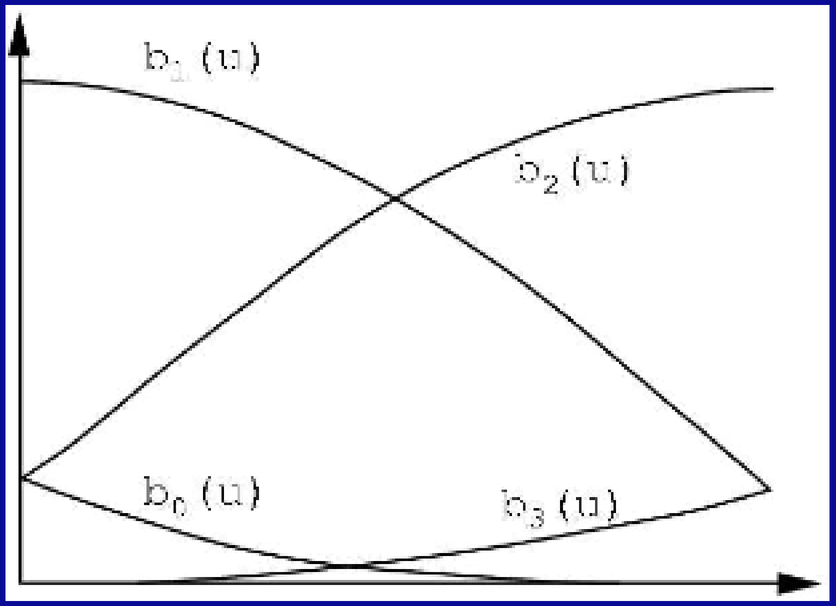



# B-Spline Basis*

- We always need 3 more control points than spline pieces

$$M_{Bs} = \frac{1}{6}\begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{bmatrix}$$

$$G_{Bsi} = \begin{bmatrix} P_{i-3} \\ P_{i-2} \\ P_{i-1} \\ P_i \end{bmatrix}$$



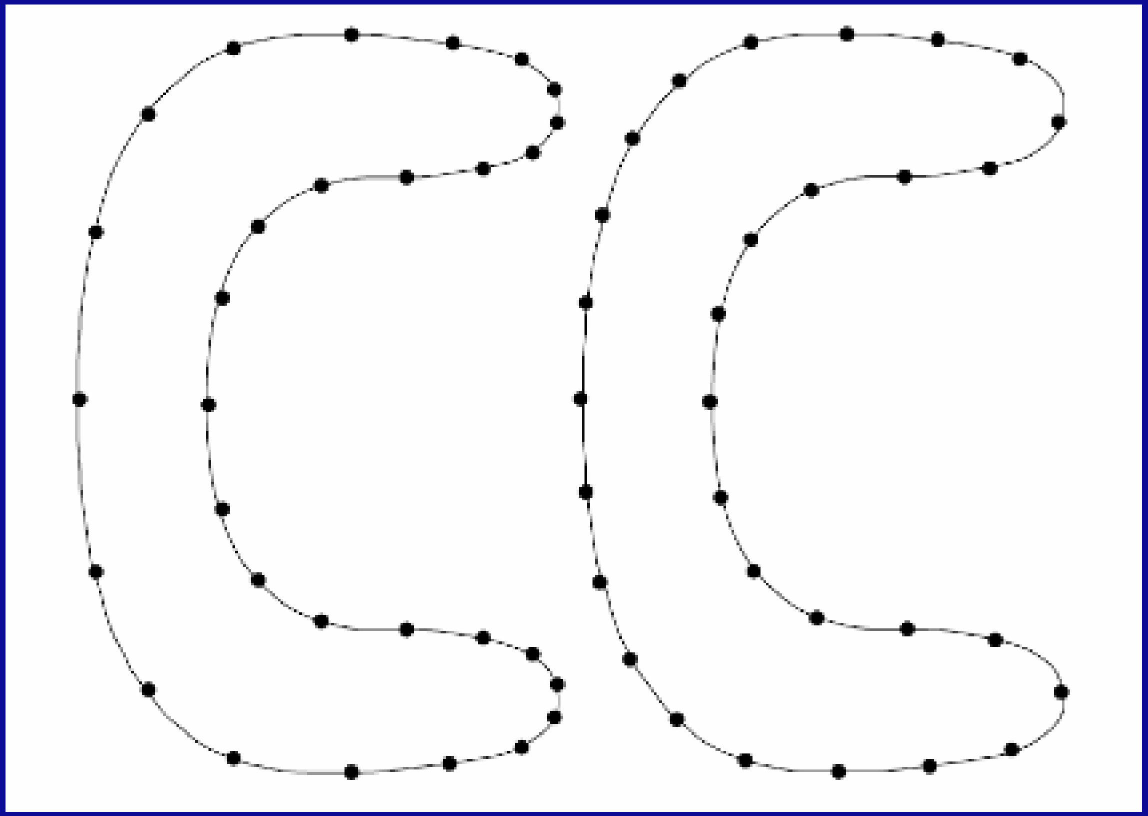
# How to Draw Spline Curves

- Basis matrix eqn. allows same code to draw any spline type
- Method 1: brute force
  - Calculate the coefficients
  - For each cubic segment, vary *u* from *0* to *1* (fixed step size)
  - Plug in *u* value, matrix multiply to compute position on curve
  - Draw line segment from last position to current position

$$\begin{bmatrix} x & y & z \end{bmatrix} = \begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix} \underbrace{\begin{bmatrix} -s & 2-s & s-2 & s \\ 2s & s-3 & 3-2s & -s \\ -s & 0 & s & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}}_{\text{CR basis}} \underbrace{\begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \\ x_4 & y_4 & z_4 \end{bmatrix}}_{\text{control vector}}$$

# How to Draw Spline Curves

- What's wrong with this approach?
  - –Draws in even steps of u
  - –Even steps of u $\neq$ even steps of x
  - –Line length will vary over the curve
  - –Want to bound line length
    - »too long: curve looks jagged
    - »too short: curve is slow to draw

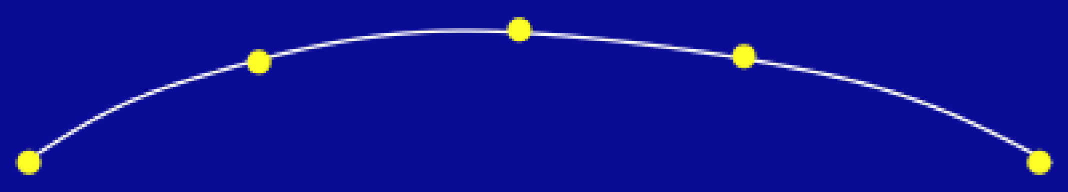


# Drawing Splines, 2

- **Method 2: recursive subdivision - vary step size to draw short lines**

```
Subdivide(u0,u1,maxlinelength)
    umid = (u0 + u1)/2
    x0 = P(u0)
    x1 = P(u1)
    if |x1 - x0| > maxlinelength
       Subdivide(u0,umid,maxlinelength)
       Subdivide(umid,u1,maxlinelength)
    else drawline(x0,x1)
```

- **Variant on Method 2 - subdivide based on curvature**
  - **replace condition in "if" statement with straightness criterion**
  - **draws fewer lines in flatter regions of the curve**



# In Summary...

- **Summary:**
  - piecewise cubic is generally sufficient
  - define conditions on the curves and their continuity

- **Things to know:**
  - basic curve properties (what are the conditions, controls, and properties for each spline type)
  - generic matrix formula for uniform cubic splines x(u) = uBG
  - given definition derive a basis matrix

# Practice Problems

Write the equation for Catmull-Rom splines in matrix form, assuming that s=1. Label the geometry (basis) matrix and the control variables.

How do you guarantee C0 continuity between two adjacent Catmull-Rom splines? C1 continuity? Give an example of control points for two adjacent curves which have C0 and C1 continuity.

# Practice Problems

There are a variety of different types of curves mentioned in these slides (Line segments, Hermite splines, Bezier splines, Catmull-Rom splines, Natural Cubic Splines, and B-splines) For each type of curve mentioned in these slides, list pros and cons of using that curve to create animations.