15-462 Spring 2012
Homework #1 Solutions

1) $a = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$ $b = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $p(t) = a + t(b-a)$

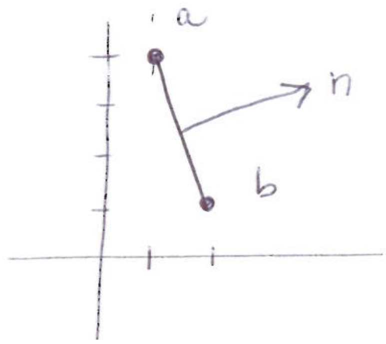
$$p(t) = \begin{bmatrix} 1 \\ 4 \end{bmatrix} + t \begin{bmatrix} 2-1 \\ 1-4 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix} + t \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

$$\boxed{p(t) = \begin{bmatrix} 1 \\ 4 \end{bmatrix} + t \begin{bmatrix} 1 \\ -3 \end{bmatrix}}$$

Check solution:

$\checkmark$ $p(0) = \begin{bmatrix} 1 \\ 4 \end{bmatrix} = a$ $\checkmark$ $p(1) = \begin{bmatrix} 1+1 \\ 4-3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} = b$

2)



Step #1: find a normal direction

$$b - a = \begin{bmatrix} 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

Rotate $b-a$ by 90° (taking $x \to y$ and $y \to -x$)

$$\boxed{n = \begin{bmatrix} 3 \\ 1 \end{bmatrix}} \Leftarrow$$ normal direction (we do not care about its magnitude)

Implicit expression for the line:

$$\boxed{(p - a) \cdot n = 0} \Leftarrow$$ This answer is fine

SIMPLE FORM

In more detail:

$$\left(\begin{bmatrix} p_x \\ p_y \end{bmatrix} - \begin{bmatrix} 1 \\ 4 \end{bmatrix} \right) \cdot \begin{bmatrix} 3 \\ 1 \end{bmatrix} = 0$$

$$\begin{bmatrix} p_x \\ p_y \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 1 \end{bmatrix} = 0$$

$$3p_x + p_y - (3+4) = 0$$

Checking our answer:

Check a:

$3(1) + 4 - 7 = 0$ ✓

Check b:

$3(2) + 1 - 7 = 0$ ✓

ok!

$$\boxed{3p_x + p_y - 7 = 0}$$

ALTERNATE FORM

3) Map $\begin{bmatrix} 2 \\ 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ AND $\begin{bmatrix} 3 \\ -2 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

We know that:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow$$

$$\boxed{\begin{aligned} 2a + 3b &= 1 \\ 2c + 3d &= 0 \\ 3a - 2b &= 0 \\ 3c - 2d &= 1 \end{aligned}}$$

We can solve these 4 equations for a, b, c, & d.

Alternatively, recall that transformations between coordinate frames can be created by placing scaled coordinate directions along the matrix rows / columns.

Solution: $\boxed{\begin{bmatrix} 2/13 & 3/13 \\ 3/13 & -2/13 \end{bmatrix}}$

← notice this row is a scaled version of u

← this row is a scaled version of v

Checking our answer:

$$\begin{bmatrix} 2/13 & 3/13 \\ 3/13 & -2/13 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$ OK

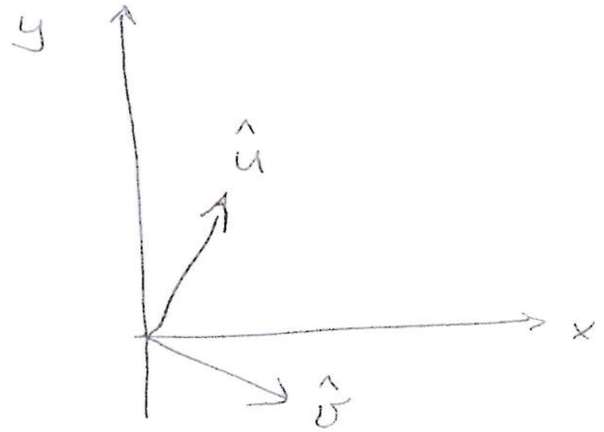
$$\begin{bmatrix} 2/13 & 3/13 \\ 3/13 & -2/13 \end{bmatrix}\begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$ OK

4.) Rows (cols) are not unit vectors FAIL

$u \cdot v = 0$ This part is OK

$u \times v \neq 1\hat{z}$ $(u \times v) = \frac{-13}{13^2} = -\frac{1}{13}$ FAIL

This is $\boxed{\text{not a rotation matrix}}$
because it also scalls the vectors
and because it represents a Mirroring action,
not a pure rotation.



$\Leftarrow$ Note that no pure rotation can do this alignment

$\underline{x \to u}$ & $\underline{y \to v}$

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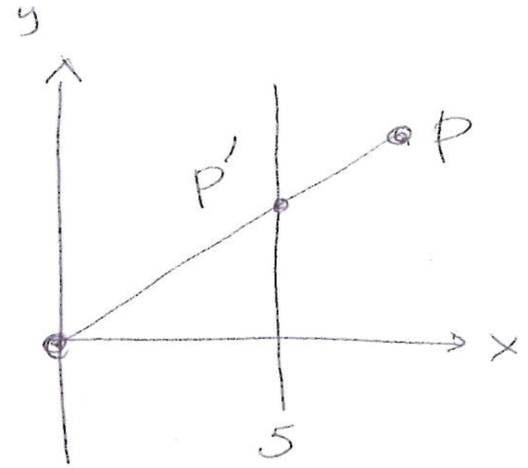
5)



From similar triangles:

$$P_y / P_x = P_y' / 5$$

$$P_y' = 5P_y / P_x$$

Find the matrix that maps

$$\begin{bmatrix} P_x \\ P_y \\ 1 \end{bmatrix} \to \begin{bmatrix} 5 \\ 5P_y / P_x \\ 1 \end{bmatrix} \propto \begin{bmatrix} 5P_x \\ 5P_y \\ P_x \end{bmatrix}$$

$$\begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ 1 \end{bmatrix} = \begin{bmatrix} 5p_x \\ 5p_y \\ p_x \end{bmatrix} \Rightarrow \begin{bmatrix} 5 \\ 5p_y/p_x \\ 1 \end{bmatrix}$$

OK

6)

Mesh (+)	Implicit Surface (+)
• Easy to render with traditional graphics pipeline	• Compact representation
• Can easily generate sharp features	• Efficient to compute distance from the surface (eg, for collision detection)
• Easy to create using traditional 3D modeling tools	• Generates pleasing smooth shapes
	• Lends itself to novel modeling interfaces (eg, sculpting as with clay)

etc......

7) Properties of the Material:

K_a · ambient reflection coefficient
(how much the material appears to "glow")

K_d · diffuse reflection coefficient
(how much light the material scatters back in a direction-independent manner)

K_s · specular reflection coefficient
(how much light the material reflects in directions very near the direction of perfect reflection)

α · specular reflection exponent
(controls sharpness of the specular reflection)

8) From Lecture 8

$$\boxed{\text{BRDF: } f(\theta_i, \phi_i; \theta_r, \phi_r) = \frac{L^{\text{surface}}(\theta_r, \phi_r)}{E^{\text{surface}}(\theta_i, \phi_i)}}$$

THE RATIO OF LIGHT <u>OUT</u> TO LIGHT <u>IN</u>, PARAMETERIZED BY RESPECTIVE DIRECTIONS

θ_i, ϕ_i parameterize the <u>incoming</u> light direction

θ_r, ϕ_r parameterize the <u>outgoing</u> light direction

$L^{\text{surface}}(\theta_r, \phi_r)$ is intensity of light travelling in the <u>outgoing</u> light direction

$E^{\text{surface}}(\theta_i, \phi_i)$ is intensity of light reaching the surface in the <u>incoming</u> direction

9) A BRDF can capture:

- Directional differences - if a surface reflects light more strongly in one direction than another. This effect is apparent in some types of cloth, due to directional differences in the weave
- Fresnel Reflectance - where an object appears More reflective as the ~~the~~ viewing angle approaches 90° (i.e., at grazing angles)
- Iridescence - where a surface appears to change color with ~~the~~ changing direction of incoming light or with viewing direction

etc...

10) $p(u) = P_1 + P_2(u-1)u + (P_3 - P_1)u$

$$= P_1(1-u) + P_2(-u+u^2) + P_3(u)$$

$$\boxed{= \begin{bmatrix} u^2 & u & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ -1 & -1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix}}$$

11.) $P_A(1) = P_B(0)$

$P_A(1) = P_{3,A}$ $\qquad$ $P_B(0) = P_{1,B}$

$$\boxed{P_{3,A} = P_{1,B}}$$

12) $P_A'(1) = P_B'(0)$

$p'(u) = P_1(-1) + P_2(-1+2u) + P_3(1)$

$P_A'(1) = -P_{1,A} + P_{2,A} + P_{3,A}$

$P_B'(0) = -P_{1,B} - P_{2,B} + P_{3,B}$

$$\boxed{-P_{1,A} + P_{2,A} + P_{3,A} = -P_{1,B} - P_{2,B} + P_{3,B}}$$