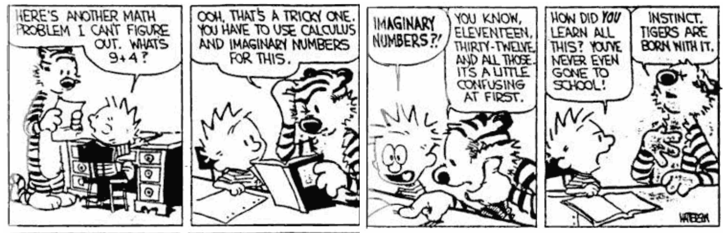
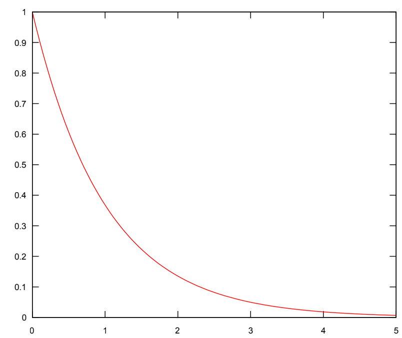
15-462: Computer Graphics
Homework 4



(Background) Consider the differential equation:

$$\dot{z} = -z,$$
$$z(0) = 1.$$



Of course, we happen to know the *exact* solution to this equation:

$$z(t) = e^{-t}.$$

In other words, z will be driven exponentially towards the origin. However, assume instead that we wanted to *simulate* this system numerically with timestep $\Delta t > 0$ using a differential equation solver, which will produce an approximating sequence:

$$z_0 = 1, z_{\Delta t}, z_{2\Delta t}, z_{3\Delta t}, \ldots$$

For Runge-Kutta methods, this sequence can be written recursively as:

$$z_{t+\Delta t} = z_t \, g(\Delta t)$$

For example, for 1st-order Runge-Kutta *(Euler's method: Lecture 24 / Slide 12)*:

$$z_{t+\Delta t} = z_t + \Delta t \, \dot{z}_t = z_t + \Delta t \, (-z_t) = z_t (1 - \Delta t)$$
$$g(\Delta t) = 1 - \Delta t$$

(1) Determine $g(\Delta t)$ for 2nd-order Runge-Kutta *(the midpoint method: Lecture 24 / Slide 21)*.

(2) Given the behavior of this system, we know that our solver will be stable and non-oscillatory as long as each element in the sequence is closer to the x-axis than its predecessor: $0 < z_{t+\Delta t} < z_t$, or equivalently:

$$0 < g(\Delta t) < 1$$

Determine the region of stability, non-oscillatory behavior for Δt for both 1st- and 2nd-order Runge-Kutta.

(3) Do these stability criteria depend on this initial condition $z(0) = 1$?