(Background) Consider the differential equation:

\[ \dot{z} = -z, \]
\[ z(0) = 1. \]

Of course, we happen to know the exact solution to this equation:

\[ z(t) = e^{-t}. \]

In other words, \( z \) will be driven exponentially towards the origin. However, assume instead that we wanted to simulate this system numerically with timestep \( \Delta t > 0 \) using a differential equation solver, which will produce an approximating sequence:

\[ z_0 = 1, z_{\Delta t}, z_{2\Delta t}, z_{3\Delta t}, \ldots \]

For Runge-Kutta methods, this sequence can be written recursively as:

\[ z_{t+\Delta t} = z_t g(\Delta t) \]

For example, for 1st-order Runge-Kutta (Euler’s method: Lecture 24 / Slide 12):

\[ z_{t+\Delta t} = z_t + \Delta t \dot{z}_t = z_t + \Delta t (-z_t) = z_t(1 - \Delta t) \]
\[ g(\Delta t) = 1 - \Delta t \]

(1) Determine \( g(\Delta t) \) for 2nd-order Runge-Kutta (the midpoint method: Lecture 24 / Slide 21).

(2) Given the behavior of this system, we know that our solver will be stable and non-oscillatory as long as each element in the sequence is closer to the \( x \)-axis than its predecessor: \( 0 < z_{t+\Delta t} < z_t \), or equivalently:

\[ 0 < g(\Delta t) < 1 \]

Determine the region of stability, non-oscillatory behavior for \( \Delta t \) for both 1st- and 2nd-order Runge-Kutta.

(3) Do these stability criteria depend on this initial condition \( z(0) = 1 \)?