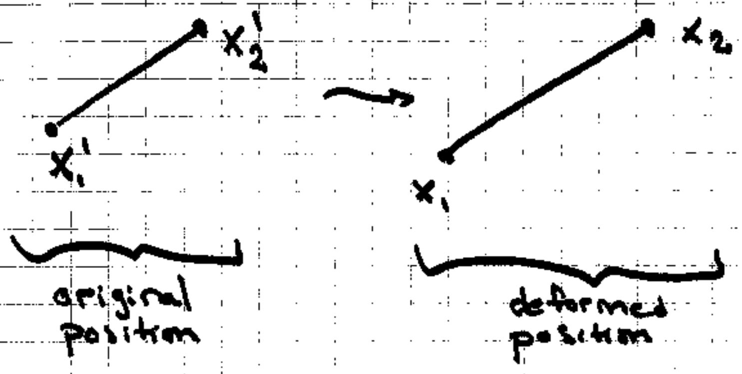
Laplacian Mesh Editing

One Edge:



We would like the relative position, $x_1 - x_2$, to remain as close as possible to the original relative position $x_1' - x_2'$:

$$f(x_1, x_2) = \frac{1}{2}\Big[\underbrace{(x_1 - x_2)}_{\text{deformed relative position}} - \underbrace{(x_1' - x_2')}_{\text{original relative position}}\Big]^2$$

NOTE BOOK

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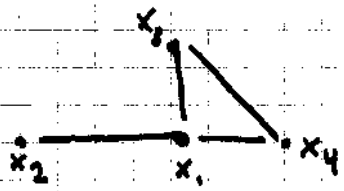
Multiple Edges:



We simply add the objective functions for each edge together:

$$
\begin{aligned}
f(x_1, x_2, x_3, x_4) = \ & \frac{1}{2}\left[(x_1 - x_2) - (x_1' - x_2')\right]^2 + \\
& \frac{1}{2}\left[(x_1 - x_3) - (x_1' - x_3')\right]^2 + \\
& \frac{1}{2}\left[(x_1 - x_4) - (x_1' - x_4')\right]^2 + \\
& \frac{1}{2}\left[(x_3 - x_4) - (x_3' - x_4')\right]^2
\end{aligned}
$$

How to find the minimum:

Set the derivative to zero: $\frac{df}{dx_1} = 0 \quad \frac{df}{dx_2} = 0 \quad \frac{df}{dx_3} = 0 \quad \frac{df}{dx_4} = 0$

$$
\begin{aligned}
\frac{df}{dx_1} = \ & (x_1 - x_2) - (x_1' - x_2') + (x_1 - x_3) - (x_1' - x_3') + \\
& (x_1 - x_4) - (x_1' - x_4') = 0
\end{aligned}
$$

$$\frac{df}{dx_2} = (x_2 - x_1) - (x_2' - x_1') = 0$$

$$\frac{df}{dx_3} = (x_3 - x_1) - (x_3' - x_1') + (x_3 - x_4) - (x_3' - x_4') = 0$$

$$\frac{df}{dx_4} = (x_4 - x_1) - (x_4' - x_1') + (x_4 - x_3) - (x_4' - x_3') = 0$$

Simplifying:

$$
\begin{aligned}
3x_1 - x_2 - x_3 - x_4 &= 3x_1' - x_2' - x_3' - x_4' \\
x_2 - x_1 &= x_2' - x_1' \\
2x_3 - x_1 - x_4 &= 2x_3' - x_1' - x_4' \\
2x_4 - x_1 - x_3 &= 2x_4' - x_1' - x_3'
\end{aligned}
$$

Written in Matrix Notation:

$$
\overbrace{\begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}}^{\text{Laplacian}}
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}
=
\begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}
\begin{bmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{bmatrix}
$$

What do we notice about the Laplacian, L:

$$
L_{ij} = \begin{cases} \text{Valence of vertex } i & \text{if } i=j \\ -1 & \text{if } i \text{ adjacent to } j \\ 0 & \text{otherwise.} \end{cases}
$$

Simplifying:

$$L\vec{x} = L\vec{x}'$$

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Constraints:

Some vertices may be constrained. Split the ~~x~~ vector into constrained & unconstrained parts: $\vec{x} = \begin{bmatrix} \vec{x}_f \\ \vec{x}_c \end{bmatrix}$ ← free / ← constrained

(A) Split the Laplacian as well

$$L\vec{x}' = \begin{bmatrix} \underset{\text{FREE}}{L_f} & \underset{\text{CONSTRAINED}}{L_c} \end{bmatrix} \cdot \begin{bmatrix} \vec{x}_f \\ \vec{x}_c \end{bmatrix} \cdots = L_f \vec{x}_f + L_c \vec{x}_c$$

(B) Move constraints to the right:

$$L_f \vec{x}_f = L\vec{x}' - L_c \vec{x}_c$$

(C) But this matrix is no longer square; we solve it by forming the normal equations (Multiplying by L_f^T):

$$\bigstar \quad \underbrace{L_f^T L_f}_{\substack{\text{Symmetric,}\\ \text{positive definite,}\\ \text{Sparse Matrix}}} \underbrace{\vec{x}_f}_{\text{unknown vector}} = \underbrace{L_f^T L \vec{x}' - L_f^T L_c \vec{x}_c}_{\text{known vector}}$$

(D) Solve the starred (★) equation