Curves and Splines
Outline

• Hermite Splines
• Catmull-Rom Splines
• Bezier Curves
• Higher Continuity: Natural and B-Splines
• Drawing Splines
Modeling Complex Shapes

- We want to build models of very complicated objects
- An equation for a sphere is possible, but how about an equation for a telephone, or a face?
- Complexity is achieved using simple pieces
  - polygons, parametric curves and surfaces, or implicit curves and surfaces
  - This lecture: parametric curves
What Do We Need From Curves in Computer Graphics?

- Local control of shape (so that easy to build and modify)
- Stability
- Smoothness and continuity
- Ability to evaluate derivatives
- Ease of rendering

Demo
Curve Usage Demo
Curve Representations

• Explicit: \( y = f(x) \)
  \[ y = mx + b \]
  – Easy to generate points
  – Must be a function: big limitation—vertical lines?
Curve Representations

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  \[ x^2 + y^2 - r^2 = 0 \]
  + Easy to test if on the curve
  - Hard to generate points
Curve Representations

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  \[ x^2 + y^2 - r^2 = 0 \]
  + Easy to test if on the curve
  - Hard to generate points

- **Parametric:** \( (x,y) = (f(u), g(u)) \)
  \[ (x, y) = (\cos u, \sin u) \]
  + Easy to generate points
Parameterization of a Curve

- *Parameterization* of a curve: how a change in $u$ moves you along a given curve in $xyz$ space.
Polynomial Interpolation

• An $n$-th degree polynomial fits a curve to $n+1$ points
  – called Lagrange Interpolation
  – result is a curve that is too wiggly, change to any control point affects entire curve (nonlocal) – *this method is poor*

• We usually want the curve to be as smooth as possible
  – minimize the wiggles
  – high-degree polynomials are bad
Linear Interpolation
Spline Interpolation
Spine Interpolation Demo
Splines: Piecewise Polynomials

• A spline is a *piecewise polynomial* - many low degree polynomials are used to interpolate (pass through) the control points

• *Cubic piecewise* polynomials are the most common:
  – piecewise definition gives local control
Piecewise Polynomials

- Spline: lots of little polynomials pieced together
- Want to make sure they fit together nicely

- $C_0$ continuity: Continuous in position
- $C_0 \& C_1$ continuity: Continuous in position and tangent vector
- $C_0 \& C_1 \& C_2$ continuity: Continuous in position, tangent, and curvature
Splines

• Types of splines:
  – Hermite Splines
  – Catmull-Rom Splines
  – Bezier Splines
  – Natural Cubic Splines
  – B-Splines
  – NURBS
Hermite Curves

- Cubic Hermite Splines

That is, we want a way to specify the end points and the slope at the end points!
Splines

chalkboard
The Cubic Hermite Spline Equation

- Using some algebra, we obtain:

\[
p(u) = \begin{bmatrix} u^3 & u^2 & u \end{bmatrix} \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ \nabla p_1 \\ \nabla p_2 \end{bmatrix}
\]

<table>
<thead>
<tr>
<th>point that gets drawn</th>
<th>basis</th>
<th>control matrix (what the user gets to pick)</th>
</tr>
</thead>
</table>

- This form typical for splines
  - basis matrix and meaning of control matrix change with the spline type
The Cubic Hermite Spline Equation

- Using some algebra, we obtain:

\[
p(u) = \begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ \nabla p_1 \\ \nabla p_2 \end{bmatrix}
\]

point that gets drawn

basis control matrix (what the user gets to pick)

\[
p(u) = \begin{bmatrix} 2u^3 - 3u^2 + 1 \\ -2u^3 + 3u^2 \\ u^3 - 2u^2 + u \\ u^3 - u^2 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ \nabla p_1 \\ \nabla p_2 \end{bmatrix}
\]

4 Basis Functions
Four Basis Functions for Hermite splines

\[ p(u) = \begin{bmatrix}
  2u^3 - 3u^2 + 1 \\
  -2u^3 + 3u^2 \\
  u^3 - 2u^2 + u \\
  u^3 - u^2
\end{bmatrix}^T \begin{bmatrix}
  p_1 \\
  p_2 \\
  \nabla p_1 \\
  \nabla p_2
\end{bmatrix} \]

4 Basis Functions

Every cubic Hermite spline is a linear combination (blend) of these 4 functions
Piecing together Hermite Curves

- It's easy to make a multi-segment Hermite spline
  - each piece is specified by a cubic Hermite curve
  - just specify the position and tangent at each “joint”
  - the pieces fit together with matched positions and first derivatives
  - gives C1 continuity

- The points that the curve has to pass through are called *knots* or *knot points*
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Problem with Hermite Splines?

• Must explicitly specify derivatives at each endpoint!

• How can we solve this?
Catmull-Rom Splines

• With Hermite splines, the designer must specify all the tangent vectors
• Catmull-Rom: an interpolating cubic spline with built-in $C^1$ continuity.

tangent at $p_i = s(p_{i+1} - p_{i-1})$
Catmull-Rom Splines

• With Hermite splines, the designer must arrange for consecutive tangents to be collinear, to get $C^1$ continuity. This gets tedious.

• Catmull-Rom: an interpolating cubic spline with \textit{built-in} $C^1$ continuity.
Catmull-Rom Spline Matrix

\[ p(u) = [u^3 \quad u^2 \quad u] \begin{bmatrix} -s & 2-s & s-2 & s \\ 2s & s-3 & 3-2s & -s \\ -s & 0 & s & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix} \]

- Derived similarly to Hermite
- Parameter s is typically set to s=1/2.
Catmull-Rom Spline Matrix

\[
\begin{bmatrix}
  x & y & z \\
\end{bmatrix} = 
\begin{bmatrix}
  u^3 & u^2 & u & 1
\end{bmatrix}
\begin{bmatrix}
  -s & 2s & -s - 2 & s \\
  2s & s - 3 & 3 - 2s & -s \\
  -s & 0 & s & 0 \\
  0 & 1 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
  x_1 & y_1 & z_1 \\
  x_2 & y_2 & z_2 \\
  x_3 & y_3 & z_3 \\
  x_4 & y_4 & z_4
\end{bmatrix}
\]

spline coefficients | CR basis | control vector
Catmull-Rom Splines

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Catmull-Rom Spline Matrix

\[ p(u) = \begin{bmatrix} u^3 & u^2 & u \end{bmatrix} \begin{bmatrix} -s & 2-s & s-2 & s \\ 2s & s-3 & 3-2s & -s \\ -s & 0 & s & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix} \]

- Derived similarly to Hermite
- Parameter \( s \) is typically set to \( s=1/2 \).
Cubic Curves in 3D

- Three cubic polynomials, one for each coordinate
  - \( x(u) = a_x u^3 + b_x u^2 + c_x u + d_x \)
  - \( y(u) = a_y u^3 + b_y u^2 + c_y u + d_y \)
  - \( z(u) = a_z u^3 + b_z u^2 + c_z u + d_z \)

- In matrix notation

\[
\begin{bmatrix}
  x(u) \\
y(u) \\
z(u)
\end{bmatrix}
= \begin{bmatrix}
  u^3 & u^2 & u \\
  1
\end{bmatrix}
\begin{bmatrix}
a_x & a_y & a_z \\
b_x & b_y & b_z \\
c_x & c_y & c_z \\
d_x & d_y & d_z
\end{bmatrix}
\]
Catmull-Rom Spline Matrix in 3D

\[
\begin{bmatrix}
  x(u) \\ y(u) \\ z(u)
\end{bmatrix} =
\begin{bmatrix}
  u^3 & u^2 & u & 1
\end{bmatrix}
\begin{bmatrix}
  -s & 2 - s & s - 2 & s \\
  2s & s - 3 & 3 - 2s & -s \\
  -s & 0 & s & 0 \\
  0 & 1 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
  x_1 & y_1 & z_1 \\
  x_2 & y_2 & z_2 \\
  x_3 & y_3 & z_3 \\
  x_4 & y_4 & z_4
\end{bmatrix}
\]

CR basis
control vector
Outline

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- **Bezier Curves**
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Problem with Catmull-Rom Splines?

• *No* control of derivatives at endpoints!
• How can we solve this?
  • We want something intuitive.
Beziers Curves*

• Another variant of the same game
• Instead of endpoints and tangents, four control points
  – points P0 and P3 are on the curve: \( P(u=0) = P0, \quad P(u=1) = P3 \)
  – points P1 and P2 are off the curve
  – \( P'(u=0) = 3(P1-P0), \quad P'(u=1) = 3(P3 - P2) \)

• Convex Hull property
  – curve contained within convex hull of control points
• Gives more control knobs (series of points) than Hermite
• Scale factor (3) is chosen to make “velocity” approximately constant
Bezier Spline Example
The Bezier Spline Matrix*

\[
\begin{bmatrix}
    x & y & z \\
\end{bmatrix} =
\begin{bmatrix}
    u^3 & u^2 & u & 1 \\
\end{bmatrix} \begin{bmatrix}
    -1 & 3 & -3 & 1 \\
    3 & -6 & 3 & 0 \\
    -3 & 3 & 0 & 0 \\
    1 & 0 & 0 & 0 \\
\end{bmatrix} \begin{bmatrix}
    x_1 & y_1 & z_1 \\
    x_2 & y_2 & z_2 \\
    x_3 & y_3 & z_3 \\
    x_4 & y_4 & z_4 \\
\end{bmatrix}
\]

Bezier basis  Bezier control vector
Beziers Blending Functions

Also known as the order 4, degree 3 Bernstein polynomials
Nonnegative, sum to 1
The entire curve lies inside the polyhedron bounded by the control points

\[ p(t) = \begin{bmatrix} (1-t)^3 \\ 3t(1-t)^2 \\ 3t^2(1-t) \\ t^3 \end{bmatrix}^T \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix} \]
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Piecewise Polynomials

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1. $C_0$ continuity: Continuous in position
2. $C_0$ & $C_1$ continuity: Continuous in position and tangent vector
3. $C_0$ & $C_1$ & $C_2$ continuity: Continuous in position, tangent, and curvature
Splines with More Continuity?

• How could we get $C^2$ continuity at control points?

• Possible answers:
  – Use higher degree polynomials
    degree 4 = quartic, degree 5 = quintic, ... but these get computationally expensive, and sometimes wiggly
  – Give up local control natural cubic splines
    A change to any control point affects the entire curve
  – Give up interpolation cubic B-splines
    Curve goes near, but not through, the control points
## Comparison of Basic Cubic Splines

<table>
<thead>
<tr>
<th>Type</th>
<th>Local Control</th>
<th>Continuity</th>
<th>Interpolation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hermite</td>
<td>YES</td>
<td>C1</td>
<td>YES</td>
</tr>
<tr>
<td>Bezier</td>
<td>YES</td>
<td>C1</td>
<td>YES</td>
</tr>
<tr>
<td>Catmull-Rom</td>
<td>YES</td>
<td>C1</td>
<td>YES</td>
</tr>
<tr>
<td>Natural</td>
<td>NO</td>
<td>C2</td>
<td>YES</td>
</tr>
<tr>
<td>B-Splines</td>
<td>YES</td>
<td>C2</td>
<td>NO</td>
</tr>
</tbody>
</table>

- **Summary**
  - Can’t get C2, interpolation and local control with cubics
Natural Cubic Splines*

- If you want 2nd derivatives at joints to match up, the resulting curves are called *natural cubic splines*.
- It’s a simple computation to solve for the cubics' coefficients. (See *Numerical Recipes in C* book for code.)
- Finding all the right weights is a *global* calculation (solve tridiagonal linear system).
B-Splines*

- Give up interpolation
  - the curve passes *near* the control points
  - best generated with interactive placement (because it’s hard to guess where the curve will go)
- Curve obeys the convex hull property
- C2 continuity and local control are good compensation for loss of interpolation
B-Spline Basis*

- We always need 3 more control points than spline pieces

\[
M_{BS} = \frac{1}{6} \begin{bmatrix}
-1 & 3 & -3 & 1 \\
3 & -6 & 3 & 0 \\
-3 & 0 & 3 & 0 \\
1 & 4 & 1 & 0
\end{bmatrix}
\]

\[
G_{BSi} = \begin{bmatrix}
P_{i-3} \\
P_{i-2} \\
P_{i-1} \\
P_i
\end{bmatrix}
\]

igraph showing B-spline basis functions.
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How to Draw Spline Curves

- Basis matrix eqn. allows same code to draw any spline type
- **Method 1:** brute force
  - Calculate the coefficients
  - For each cubic segment, vary $u$ from 0 to 1 (fixed step size)
  - Plug in $u$ value, matrix multiply to compute position on curve
  - Draw line segment from last position to current position

\[
\begin{bmatrix}
  x & y & z \\
\end{bmatrix} =
\begin{bmatrix}
  u^3 & u^2 & u & 1 \\
\end{bmatrix}
\begin{bmatrix}
  -s & 2-s & s-2 & s \\
  2s & s-3 & 3-2s & -s \\
  -s & 0 & s & 0 \\
  0 & 1 & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
  x_1 & y_1 & z_1 \\
  x_2 & y_2 & z_2 \\
  x_3 & y_3 & z_3 \\
  x_4 & y_4 & z_4 \\
\end{bmatrix}
\]

CR basis control vector
How to Draw Spline Curves

• What’s wrong with this approach?
  – Draws in even steps of $u$
  – Even steps of $u \neq$ even steps of $x$
  – Line length will vary over the curve
  – Want to bound line length
    » too long: curve looks jagged
    » too short: curve is slow to draw
Drawing Splines, 2

- **Method 2: recursive subdivision** - vary step size to draw short lines

  \[
  \text{Subdivide}(u_0, u_1, \text{maxlinelength})
  \]
  \[
  \text{umid} = (u_0 + u_1)/2
  \]
  \[
  x_0 = P(u_0)
  \]
  \[
  x_1 = P(u_1)
  \]
  \[
  \text{if } |x_1 - x_0| > \text{maxlinelength}
  \]
  \[
  \text{Subdivide}(u_0, \text{umid}, \text{maxlinelength})
  \]
  \[
  \text{Subdivide}(\text{umid}, u_1, \text{maxlinelength})
  \]
  \[
  \text{else drawline}(x_0, x_1)
  \]

- **Variant on Method 2** - subdivide based on curvature
  - replace condition in “if” statement with straightness criterion
  - draws fewer lines in flatter regions of the curve
In Summary...

• Summary:
  – piecewise cubic is generally sufficient
  – define conditions on the curves and their continuity

• Things to know:
  – basic curve properties (what are the conditions, controls, and properties for each spline type)
  – generic matrix formula for uniform cubic splines $x(u) = uB$G
  – given definition derive a basis matrix