

# 15-462: Computer Graphics

## Math for Computer Graphics

# Announcements!

**Reading:** OpenGL Red Book Chapters 1 & 2  
**Project 1** Assigned Next Tuesday

# Topics for Today

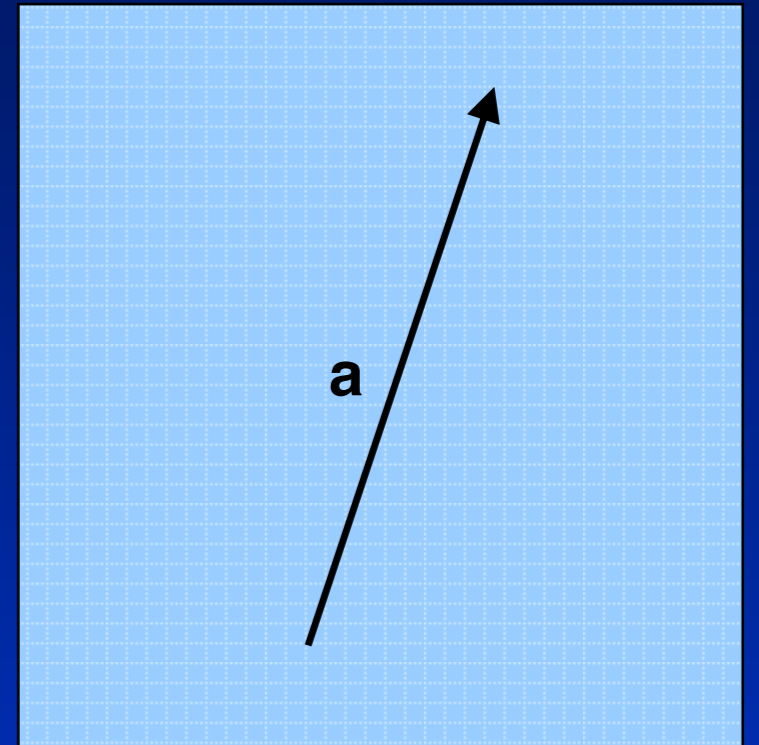
- Vectors
- Equations for curves and surfaces
- Barycentric Coordinates

# Topics for Today

- Vectors
  - What is a vector?
  - Coordinate systems
  - Vector arithmetic
  - Dot product
  - Cross product
  - Normal vectors
- Equations for curves and surfaces
- Barycentric Coordinates

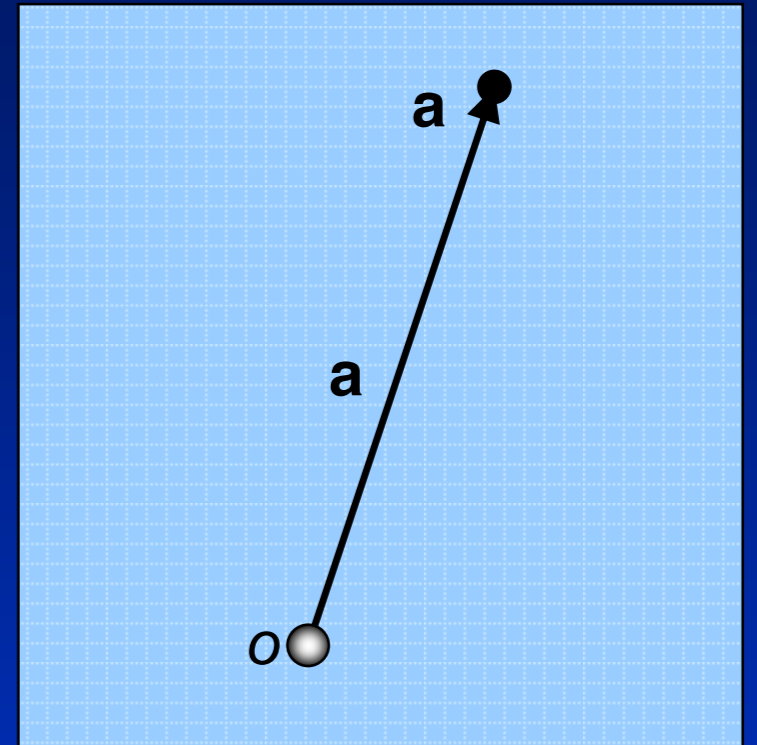
# What is a vector?

- A *vector* is a value that describes both a magnitude and a direction. We draw vectors as arrows, and name them with bold letters, e.g. **a**.



# What is a vector?

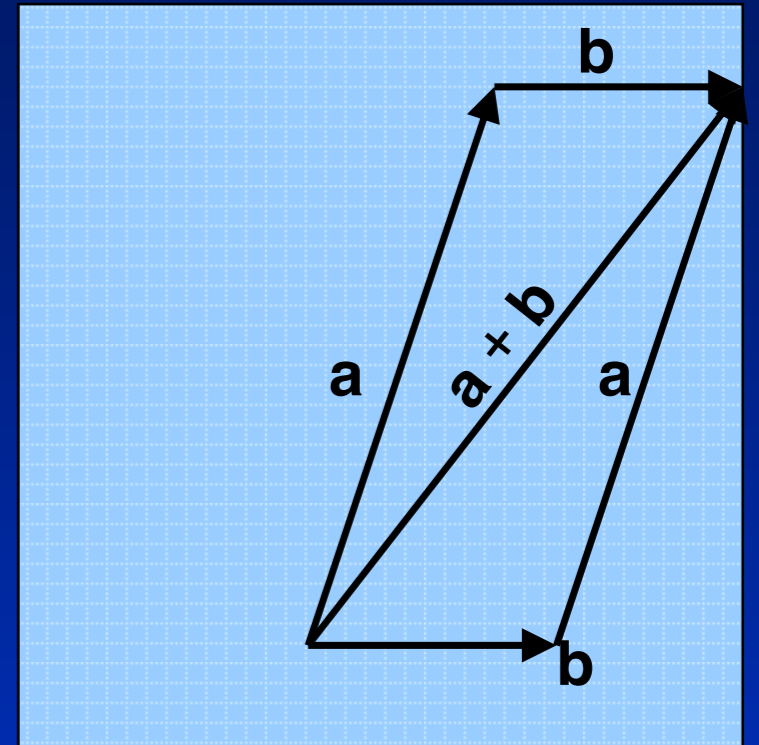
- Vectors themselves contain no information about a starting point.
- We can interpret vectors as *displacements*, instructions to get from one point in space to another.
- We can also interpret vectors as *points*, but in order to do so, we must assume a particular *origin* as the starting point.



# Vector arithmetic

- To find the *sum* of two vectors, we place the tail of one to the head of the other. The sum is the vector that completes the triangle.
- Vector addition is commutative:

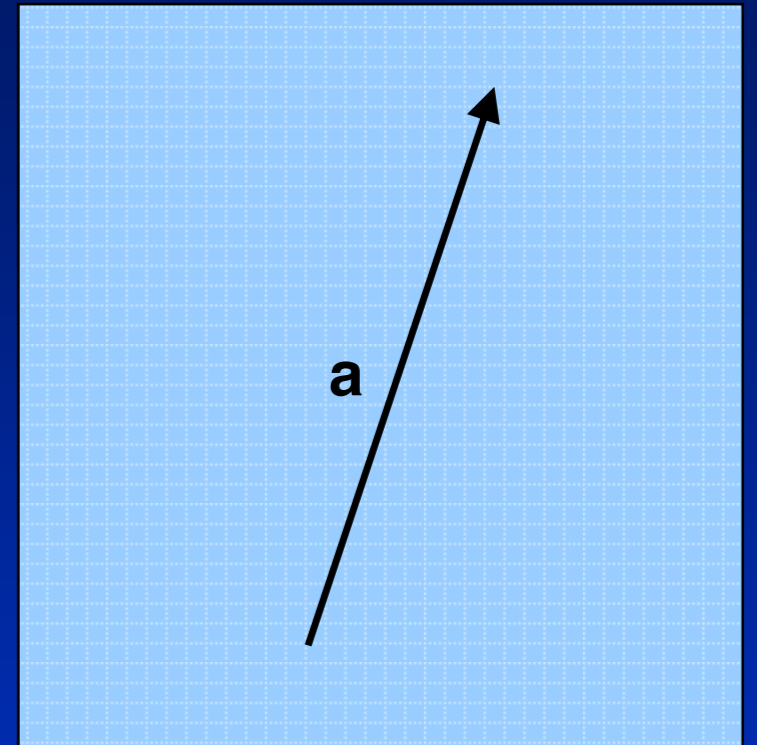
$$\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$$



# What is a vector?

## Some Definitions

- The *magnitude* of vector  $\mathbf{a}$  is the scalar given by  $\|\mathbf{a}\|$ .
- A *unit vector* is any vector whose magnitude is one.
- The *zero vector*,  $\mathbf{0}$ , has a magnitude of zero, and its direction is undefined.
- Two vectors are equal if and only if they have equal magnitudes and point in the same direction.





# Vector arithmetic

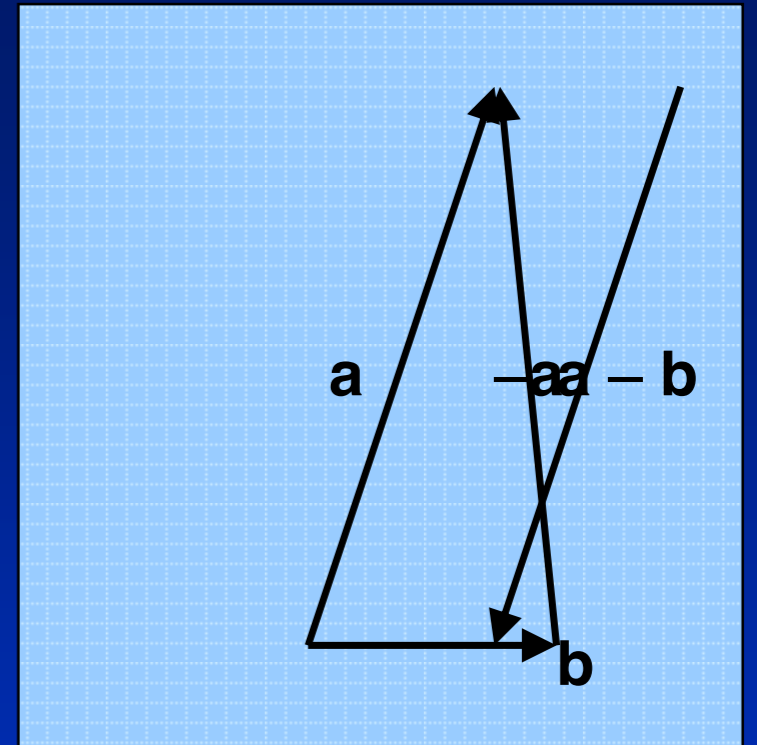
- We define the *unary minus* (negative) such that

$$-a + a = 0$$

- We can then define *subtraction* as

$$a - b \equiv -b + a$$

- This gives the vector from the end of **b** to the end of **a** if both have the same origin.

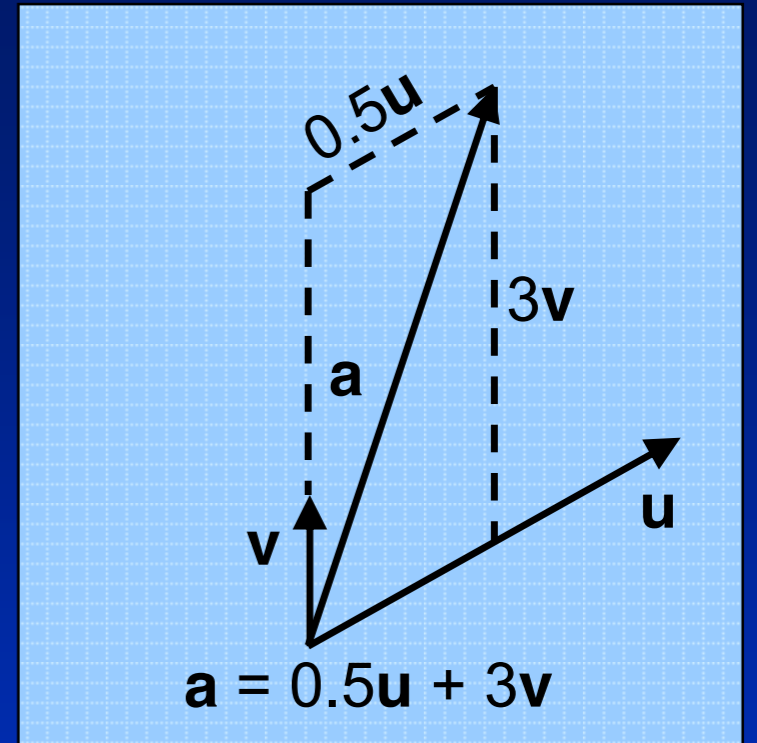


# Coordinate systems

- A vector can be multiplied by a scalar to scale the vector's magnitude without changing its direction:

$$\|k\mathbf{a}\| = k\|\mathbf{a}\|$$

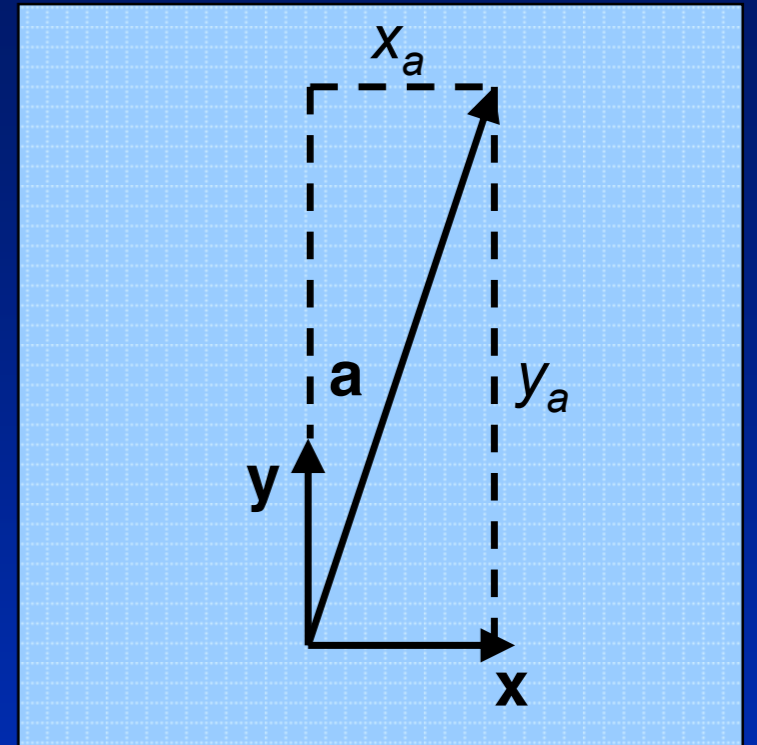
- In 2D, we can represent any vector as a unique *linear combination*, or weighted sum, of any two non-parallel *basis vectors*.
- 3D requires three non-parallel, non-coplanar basis vectors.



# Coordinate systems

- Basis vectors that are unit vectors at right angles to each other are called *orthonormal*.
- The **x-y Cartesian** coordinate system is a special orthonormal system.
- Vectors are commonly represented in terms of their Cartesian coordinates:

$$\mathbf{a} = (x_a, y_a) \quad \mathbf{a} = \begin{bmatrix} x_a \\ y_a \end{bmatrix} \quad \mathbf{a}^T = [x_a \quad y_a]$$



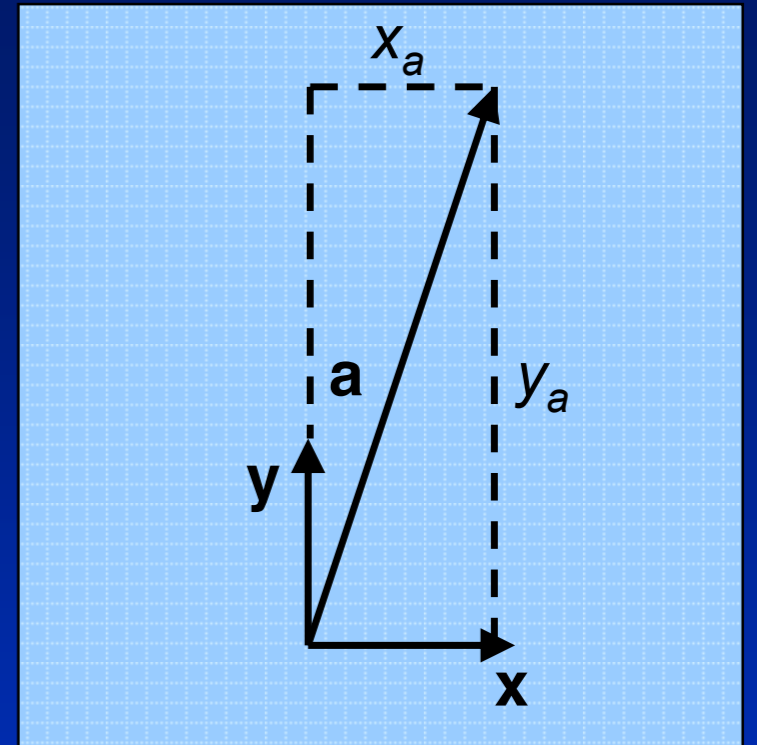
# Coordinate systems

- Vectors expressed by orthonormal coordinates

$$\mathbf{a} = (x_a, y_a)$$

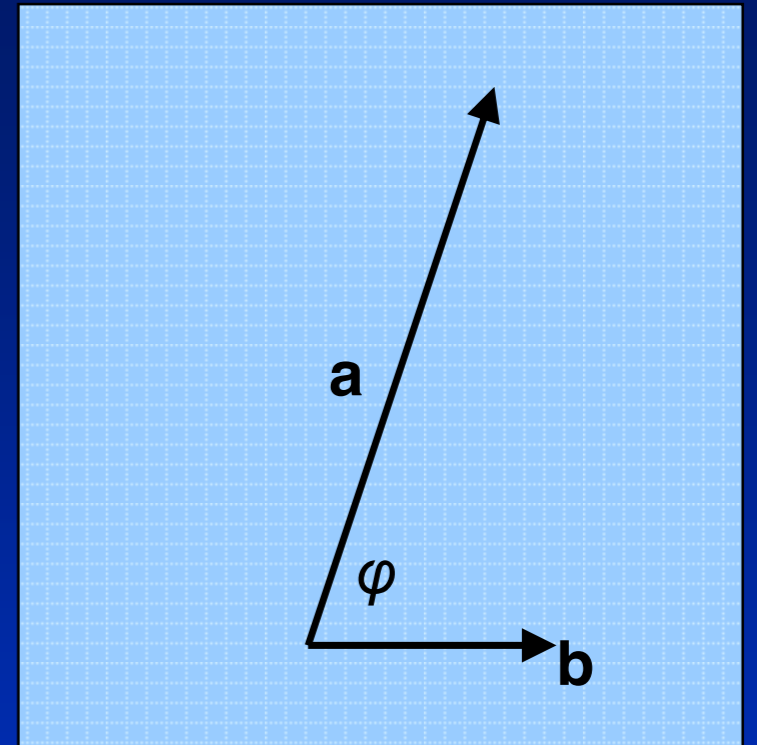
have the very useful property that their magnitudes can be calculated according to the Pythagorean Theorem:

$$\|\mathbf{a}\| = \sqrt{x_a^2 + y_a^2}$$



# Dot product

- We can multiply two vectors by taking the *dot product*.
- The dot product is defined as
$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \|\mathbf{b}\| \cos \varphi$$
where  $\varphi$  is the angle between the two vectors.
- Note that the dot product takes two vectors as arguments, but it is often called the *scalar product* because its result is a scalar.



# Dot product

## Some cool properties:

- It's often useful in graphics to know the cosine of the angle between two vectors, and we can find it with the dot product:

$$\cos \varphi = \mathbf{a} \cdot \mathbf{b} / (||\mathbf{a}|| ||\mathbf{b}||)$$

- We can use the dot product to find the *projection* of one vector onto another. The scalar  $\mathbf{a} \rightarrow \mathbf{b}$  is the magnitude of the vector  $\mathbf{a}$  projected at a right angle onto vector  $\mathbf{b}$ , and

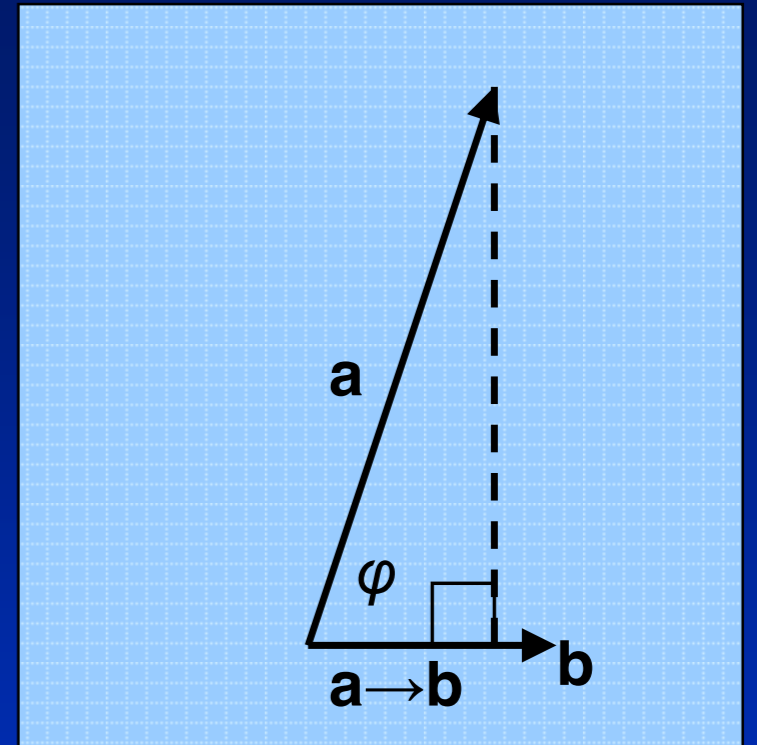
$$\mathbf{a} \rightarrow \mathbf{b} = ||\mathbf{a}|| \cos \varphi = \mathbf{a} \cdot \mathbf{b} / ||\mathbf{b}||$$

- Dot products are commutative and distributive:

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$$

$$\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$$

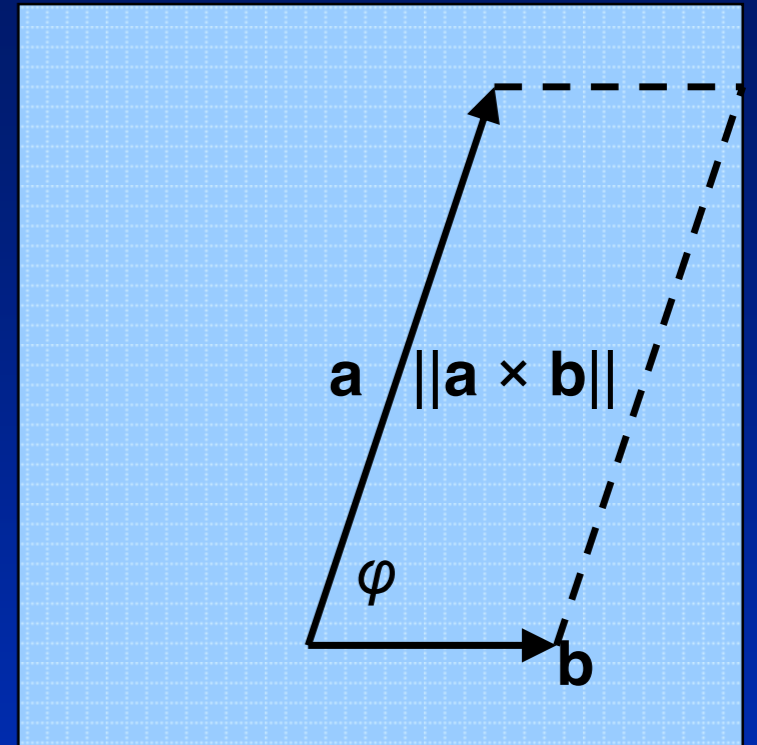
$$(k\mathbf{a}) \cdot \mathbf{b} = \mathbf{a} \cdot (k\mathbf{b}) = k(\mathbf{a} \cdot \mathbf{b})$$



# Cross product

- The *cross product* is another vector multiplication operation, usually used only for 3D vectors.
- The direction of  $\mathbf{a} \times \mathbf{b}$  is orthogonal to both  $\mathbf{a}$  and  $\mathbf{b}$ .
- The magnitude is equal to the area of the parallelogram formed by the two vectors. It is given by

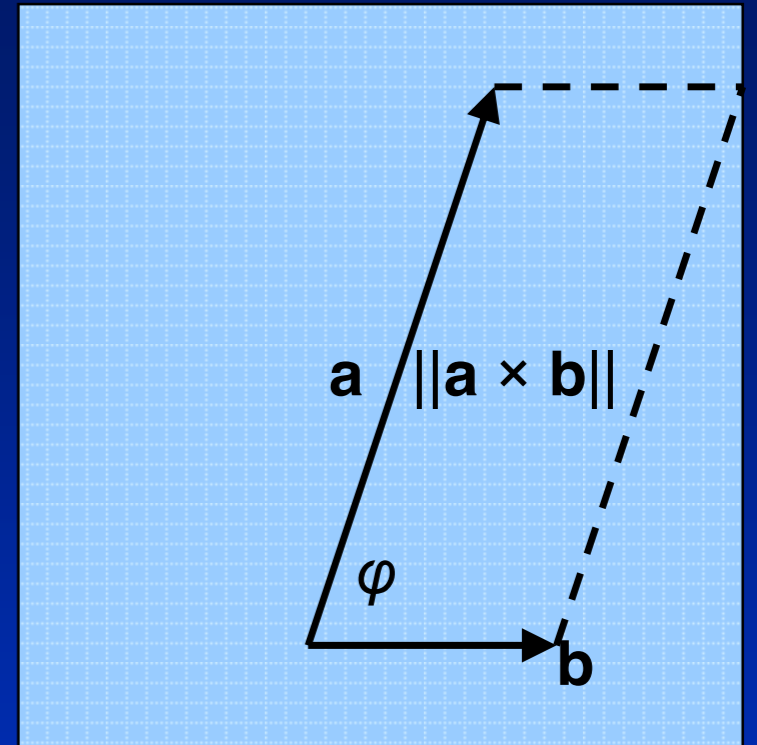
$$\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\| \|\mathbf{b}\| \sin \varphi$$



# Cross product

## Some cool properties:

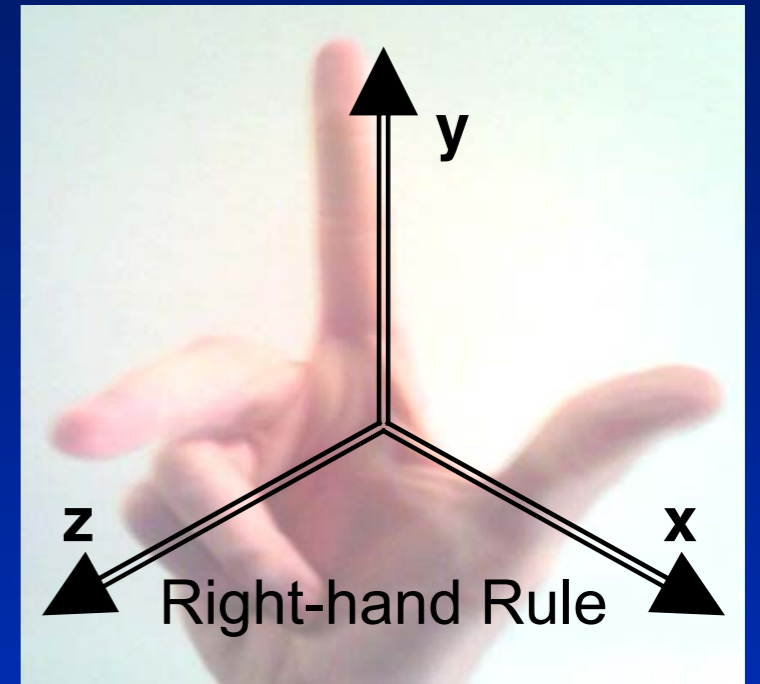
- Cross products are distributive:  
$$\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$$
$$(k\mathbf{a}) \times \mathbf{b} = \mathbf{a} \times (k\mathbf{b}) = k(\mathbf{a} \times \mathbf{b})$$
- Cross products are intransitive; in fact,  
$$\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$$
- Because of the sine in the magnitude calculation, for all  $\mathbf{a}$ ,  
$$\mathbf{a} \times \mathbf{a} = \mathbf{0}$$
- In  $\mathbf{x}$ - $\mathbf{y}$ - $\mathbf{z}$  Cartesian space,  
$$\mathbf{x} \times \mathbf{y} = \mathbf{z} \quad \mathbf{y} \times \mathbf{z} = \mathbf{x} \quad \mathbf{z} \times \mathbf{x} = \mathbf{y}$$





# Cross product

- As defined on previous slides, the direction of the cross product is ambiguous.
- The *left-hand rule* and the *right-hand rule* distinguish the two choices.
- If  $\mathbf{a}$  points in the direction of your thumb and  $\mathbf{b}$  points in the direction of your index finger,  $\mathbf{a} \times \mathbf{b}$  points in the direction of your middle finger.
- Of the two, the right-hand rule is the predominant convention.



# Normal vectors

- A *normal vector* is a vector perpendicular to a surface. A *unit normal* is a normal vector of magnitude one.
- Normal vectors are important to many graphics calculations.
- If the surface is a polygon containing the points **a**, **b**, and **c**, one normal vector
$$\mathbf{n} = (\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{a})$$
- This vector points *into* the polygon if **a**, **b**, and **c** are arranged clockwise; it points outward if they are arranged counterclockwise.

# Vectors

Chalkboard examples:

- Cartesian vector addition
- Cartesian dot product
- Cartesian cross product

# Topics for Today

- Vectors
- Equations for curves and surfaces
  - Implicit equations
  - Parametric equations
- Barycentric Coordinates

# Implicit equations

- *Implicit equations* are a way to define curves and surfaces.

- In 2D, a curve can be defined by

$$f(x,y) = 0$$

for some scalar function  $f$  of  $x$  and  $y$ .

- In 3D, a surface can be defined by

$$f(x,y,z) = 0$$

for some scalar function  $f$  of  $x$ ,  $y$ , and  $z$ .

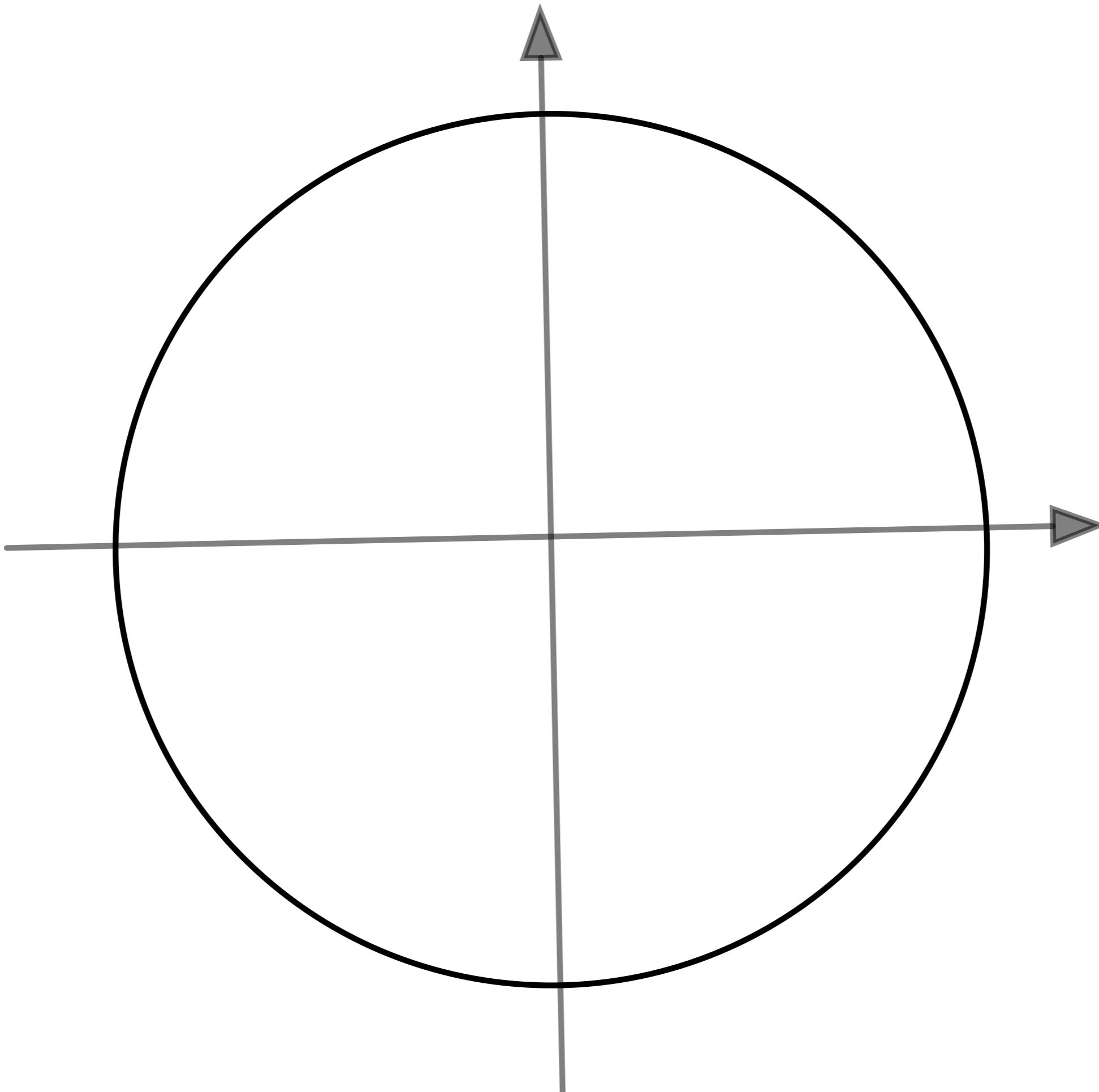
# Implicit equations

- The function  $f$  evaluates to 0 at every point on the curve or surface, and it evaluates to a non-zero real number at all other points.
- Multiplying  $f$  by a non-zero coefficient preserves this property, so we can rewrite

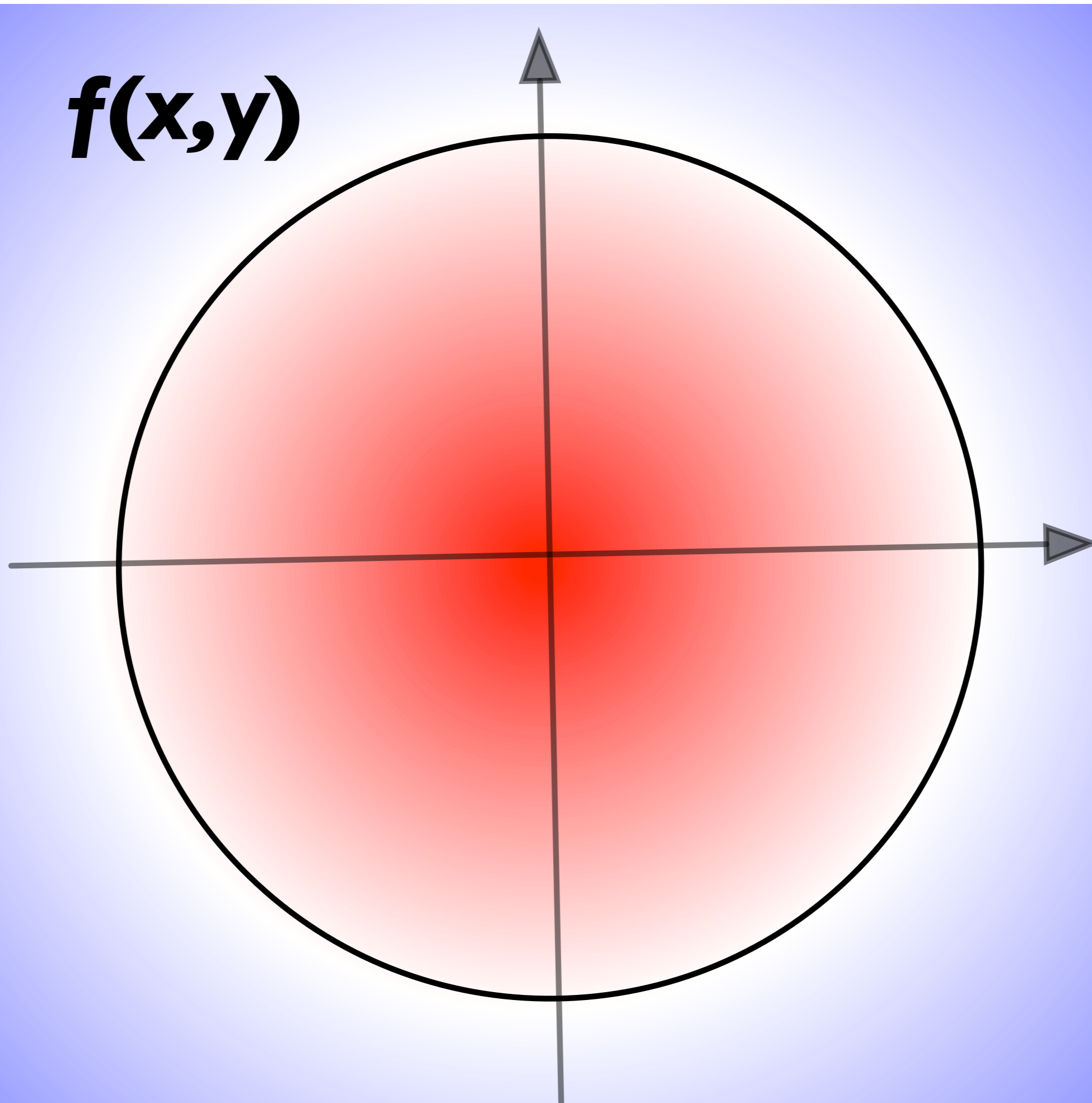
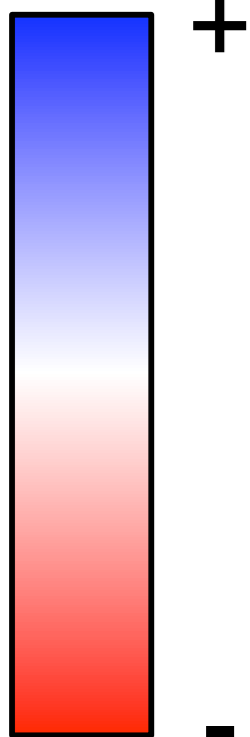
$$f(x,y) = 0$$
$$\text{as } kf(x,y) = 0$$

for any non-zero  $k$ .

- The implied curve is unaffected.



**$f(x,y)$**





# Implicit equations

Chalkboard examples:

- Implicit 2D circle
- Implicit 2D line
- Implicit 3D plane

# Implicit equations

- We call these equations “implicit” because although they imply a curve or surface, they cannot explicitly generate the points that comprise it.
- In order to generate points, we need another form...

# Parametric equations

- *Parametric equations* offer the capability to generate continuous curves and surfaces.
- For curves, parametric equations take the form

$$x = f(t) \quad y = g(t) \quad z = h(t)$$

- For 3D surfaces, we have

$$x = f(s,t) \quad y = g(s,t) \quad z = h(s,t)$$

# Parametric equations

- The *parameters* for these equations are scalars that range over a continuous (possibly infinite) interval.
- Varying the parameters over their entire intervals smoothly generates every point on the curve or surface.

# Implicit equations

Chalkboard examples:

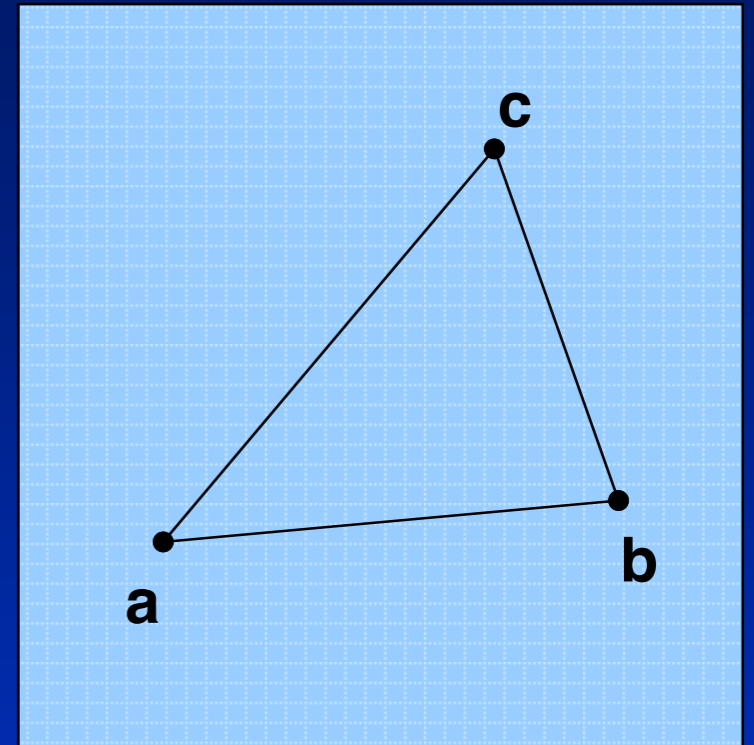
- Parametric 3D line
- Parametric sphere

# Topics for Today

- Vectors
- Equations for curves and surfaces
- Barycentric Coordinates
  - Why barycentric coordinates?
  - What are barycentric coordinates?

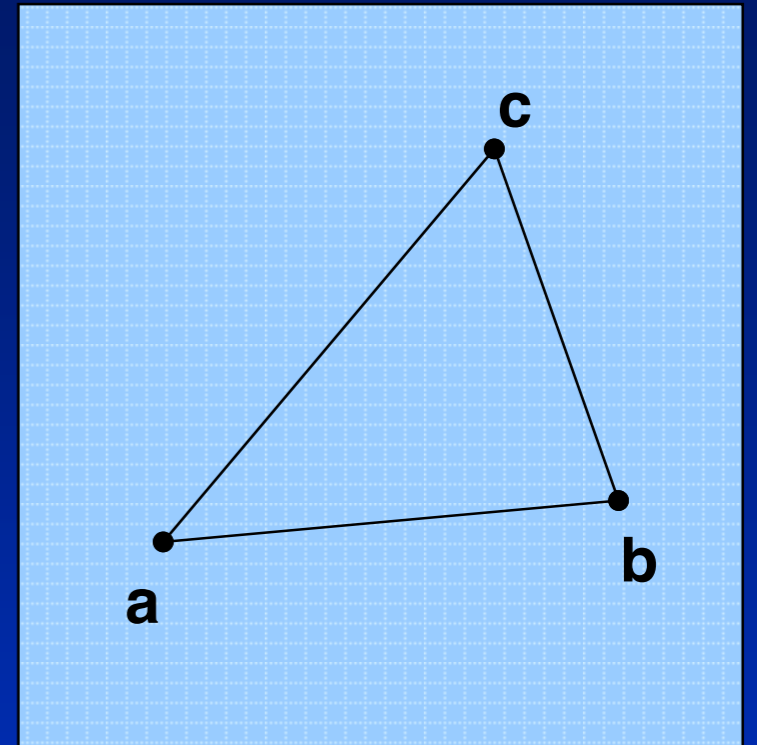
# Why barycentric coordinates?

- Triangles are the fundamental primitive used in 3D modeling programs.
- Triangles are stored as a sequence of three vectors, each defining a vertex.
- Often, we know information about the vertices, such as color, that we'd like to interpolate over the whole triangle.



# What are barycentric coordinates?

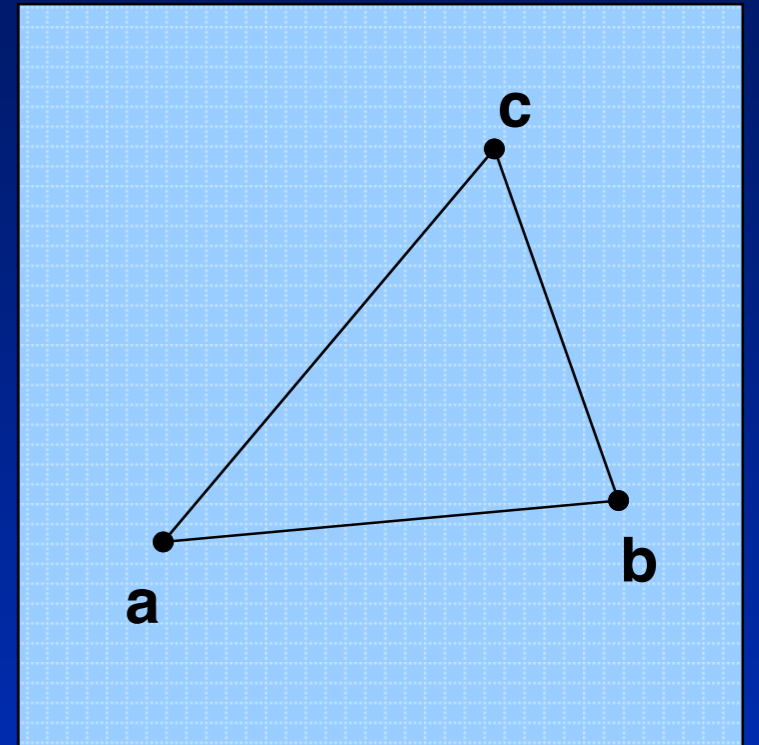
- The simplest way to do this interpolation is *barycentric coordinates*.
- The name comes from the Greek word *barus* (heavy) because the coordinates are weights assigned to the vertices.
- Point **a** on the triangle is the origin of the non-orthogonal coordinate system.
- The vectors from **a** to **b** and from **a** to **c** are taken as basis vectors.





# What are barycentric coordinates?

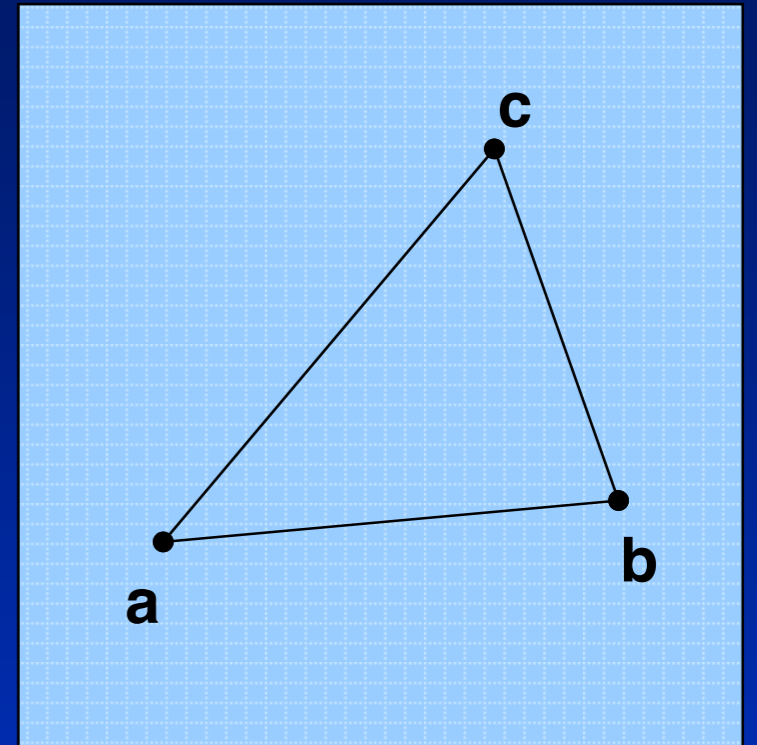
- We can express any point  $\mathbf{p}$  coplanar to the triangle as:  
$$\mathbf{p} = \mathbf{a} + \beta(\mathbf{b} - \mathbf{a}) + \gamma(\mathbf{c} - \mathbf{a})$$
- Typically, we rewrite this as:  
$$\mathbf{p}(\alpha, \beta, \gamma) = \alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c}$$
where  $\alpha \equiv 1 - \beta - \gamma$
- $\mathbf{a} = \mathbf{p}(1, 0, 0)$ ,  $\mathbf{b} = \mathbf{p}(0, 1, 0)$ ,  
 $\mathbf{c} = \mathbf{p}(0, 0, 1)$



# What are barycentric coordinates?

## Some cool properties:

- Point  $p$  is inside the triangle if and only if
$$0 < \alpha < 1,$$
$$0 < \beta < 1,$$
$$0 < \gamma < 1$$
- If one component is zero,  $p$  is on an edge.
- If two components are zero,  $p$  is on a vertex.
- The coordinates can be used as weighting factors for properties of the vertices, like color.



# Barycentric coordinates

Chalkboard examples:

- Conversion from 2D Cartesian
- Conversion from 3D Cartesian