Problem 2: ID Convolution

Compute $c(y) = a(x) * b(x)$, i.e. the convolution of $a(x)$ and $b(x)$. Simplify your answer. Show all steps.

Solution:

First, let write $a(x)$ and $b(x)$ as piecewise function:

$$a(x) = \begin{cases} 1, & -1 \leq x \leq 1 \\ 0, & \text{else} \end{cases}$$

$$b(x) = \begin{cases} 1 + x, & -1 \leq x \leq 0 \\ 1 - x, & 0 < x \leq 1 \\ 0, & \text{else} \end{cases}$$

Now, we can compute $c(x)$ using the definition of convolution. (Typo here, it should be $c(y)$ and so all the $x$ in the final answer is actually $y$).

$$c(x) = a(x) * b(x) = \int_{-\infty}^{\infty} a(t) b(x - t) \, dt = \int_{-\infty}^{\infty} a(x - t) b(t) \, dt$$

We will be using the second form (where we flip $a(x)$ as it is considered easier to flip the less complicated function when you do convolution).

So, we now have the following for $a(x - t)$ while the $b(t)$ stays with the same shape (just that it is now a function of $t$ instead of $x$).
For convolution, it will be easier to consider by cases:

**Case 1:** When the rectangle is sliding into the first half of the triangle.

This happens when \(-2 \leq x \leq -1\)

\[
c(x) = \int_{-1}^{x+1} 1 + t \ dt
\]

Solving for the integral yields: \(c(x) = \frac{1}{2} (x + 2)^2\)

**Case 2:** When the rectangle is sliding into the second half of the triangle.

This happens when \(-1 < x \leq 0\)

\[
c(x) = \int_{-1}^{x} 1 + t \ dt + \int_{0}^{1+x} 1 - t \ dt
\]

Solving for the integral yields: \(c(x) = 1 - \frac{x^2}{2}\)

**Case 3:** When the rectangle is sliding out of the first half of the triangle.

This happens when \(0 < x \leq 1\)
Solving for the integral yields:

Case 4: When the rectangle is sliding out of the second half of the triangle.

This happens when $1 < x \leq 2$

Case 5: When the rectangle and the triangle do not overlap and thus, $c(x) = 0$.

This happens when $x < -2$ and $2 < x$.

Final answer:

$$c(x) = \begin{cases} 
\frac{1}{2}(x + 2)^2 & , -2 \leq x \leq -1 \\
1 - \frac{x^2}{2} & , -1 < x \leq 0 \\
1 + \frac{x^2}{2} & , 0 < x \leq 1 \\
\frac{1}{2}(x - 2)^2 & , 1 < x \leq 2 \\
0 & \text{otherwise}
\end{cases}$$