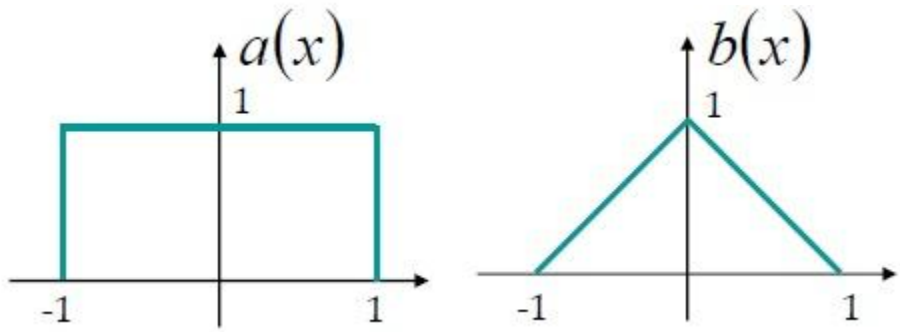
Problem 2: ID Convolution



Compute $c(y) = a(x) * b(x)$, i.e. the convolution of $a(x)$ and $b(x)$. Simplify your answer. Show all steps.

Solution:

First, let write $a(x)$ and $b(x)$ as piecewise function:

$$a(x) = \begin{cases} 1, & -1 \le x \le 1 \\ 0, & else \end{cases}$$

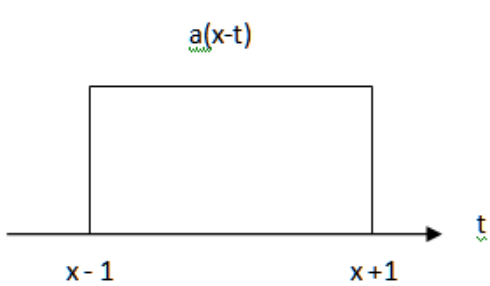
$$b(x) = \begin{cases} 1 + x, & -1 \le x \le 0 \\ 1 - x, & 0 < x \le 1 \\ 0 & else \end{cases}$$

Now, we can compute $c(x)$ using the definition of convolution. (Typo here, it should be c(y) and so all the x in the final answer is actually y).

$$c(x) = a(x) * b(x) = \int_{-\infty}^{\infty} a(t)\, b(x-t)\, dt = \int_{-\infty}^{\infty} a(x-t)\, b(t)\, dt$$

We will be using the second form (where we flip a(x) as it is considered easier to flip the less complicated function when you do convolution).

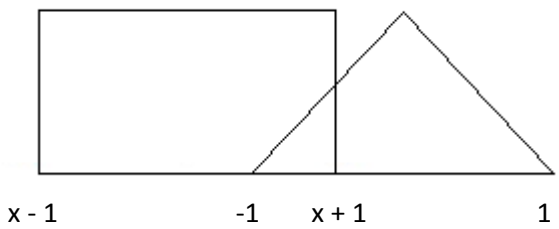
So, we now have the following for $a(x-t)$ while the $b(t)$ stays with the same shape (just that it is now a function of t instead of x.



For convolution, it will be easier to consider by cases:

Case 1: When the rectangle is sliding into the first half of the triangle.

This happens when $-2 \leq x \leq -1$

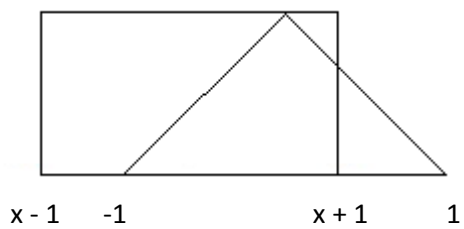


$$c(x) = \int_{-1}^{x+1} 1 + t \ dt$$

Solving for the integral yields: $c(x) = \frac{1}{2}(x+2)^2$

Case 2: When the rectangle is sliding into the second half of the triangle.

This happens when $-1 < x \leq 0$

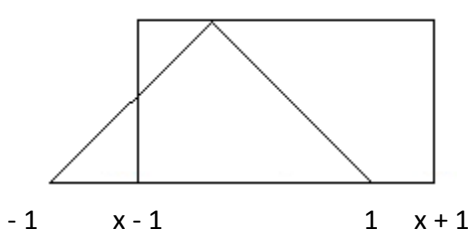


$$c(x) = \int_{-1}^{0} 1 + t \ dt + \int_{0}^{1+x} 1 - t \ dt$$

Solving for the integral yields: $c(x) = 1 - \frac{x^2}{2}$

Case 3: When the rectangle is sliding out of the first half of the triangle.

This happens when $0 < x \leq 1$

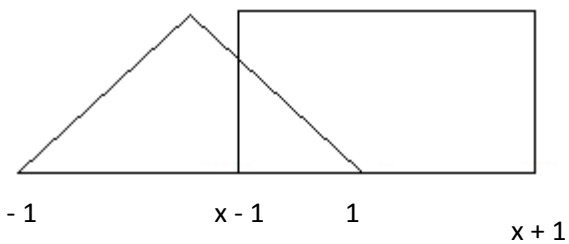


$$c(x) = \int_{-1+x}^{0} 1 + t \ dt + \int_{0}^{1} 1 - t \ dt$$

Solving for the integral yields: $c(x) = \ 1 - \frac{x^2}{2}$

Case 4: When the rectangle is sliding out of the second half of the triangle.

This happens when $1 < x \ \leq \ 2$



$$c(x) = \int_{-1+x}^{1} 1 - t \ dt$$

Solving for the integral yields: $c(x) = \frac{1}{2}(x-2)^2$

Case 5: When the rectangle and the triangle do not overlap and thus, $c(x) = \ 0$.

This happens when $x < -2$ and $2 < x$.

Final answer:

$$c(x) = \begin{cases} \frac{1}{2}(x+2)^2 & , -2 \leq x \ \leq -1 \\ 1 - \frac{x^2}{2} & , -1 < x \ \leq 0 \\ 1 - \frac{x^2}{2}, & ,0 < x \ \leq 1 \\ \frac{1}{2}(x-2)^2 & ,1 < x \ \leq 2 \\ 0 & otherwise \end{cases}$$