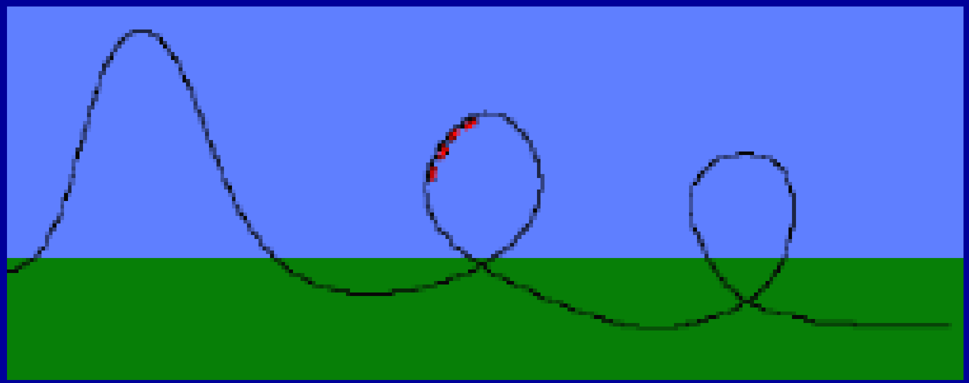
Parametric Curves

Modeling:
- parametric curves (Splines)
- polygonal meshes

Roller coaster

- Next programming assignment involves creating a 3D roller coaster animation
- We must model the 3D curve describing the roller coaster, but how?
- How to make the simulation obey the laws of gravity?

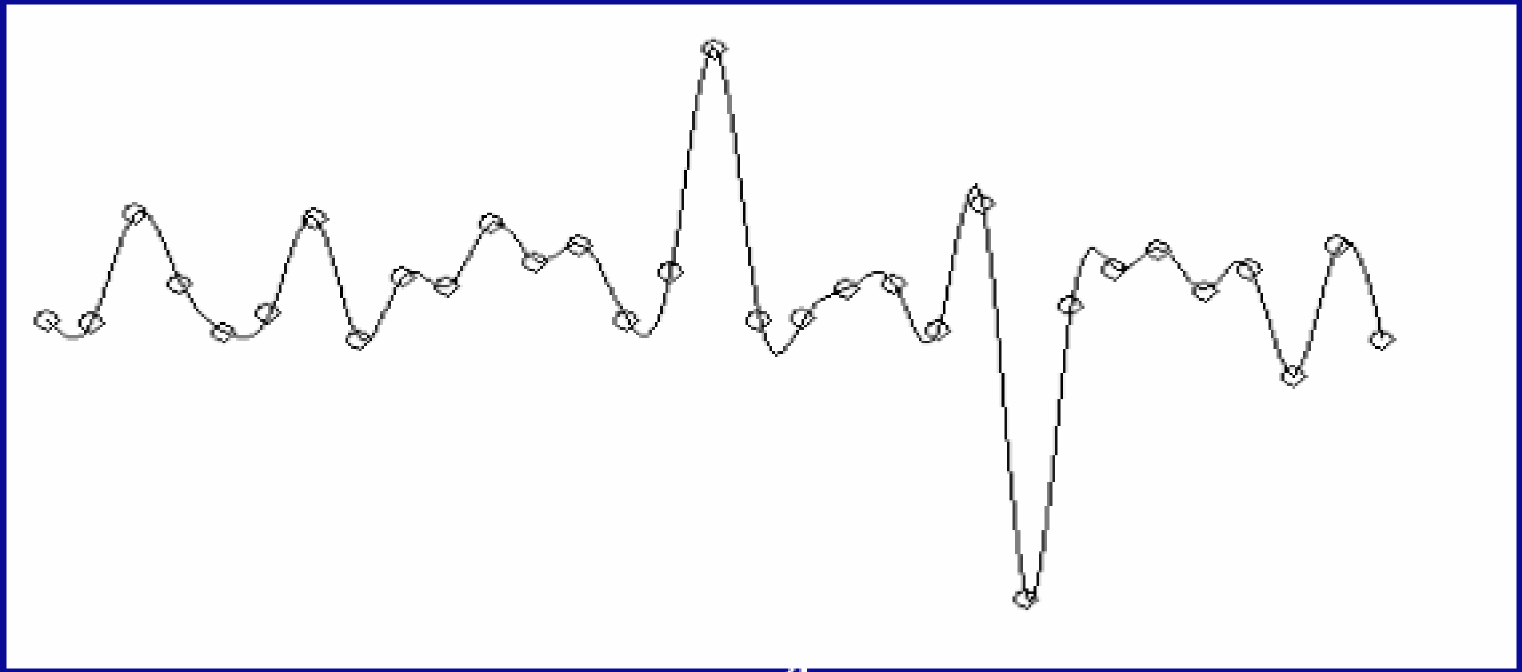


Modeling Complex Shapes

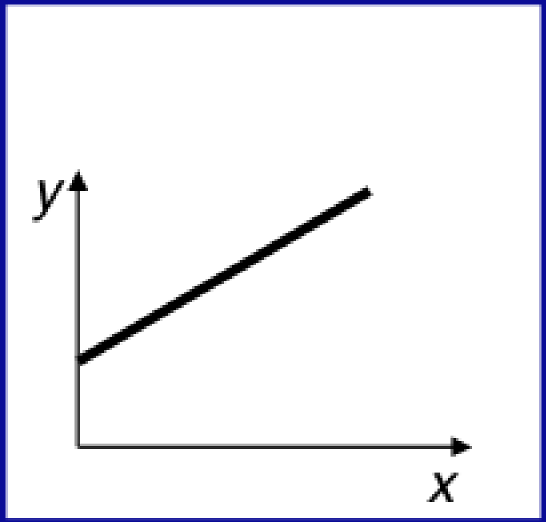
- We want to build models of very complicated objects
- An equation for a sphere is possible, but how about an equation for a telephone, or a face?
- Complexity is achieved using simple pieces
  - polygons, parametric curves and surfaces, or implicit curves and surfaces
  - This lecture: parametric curves

What Do We Need From Curves in Computer Graphics?

- Local control of shape (so that easy to build and modify)
- Stability
- Smoothness and continuity
- Ability to evaluate derivatives
- Ease of rendering



Curve Representations



- **Explicit: y = f(x)**

  $$y = mx + b$$

  - **Easy to generate points**
  - **Must be a function: big limitation—vertical lines?**

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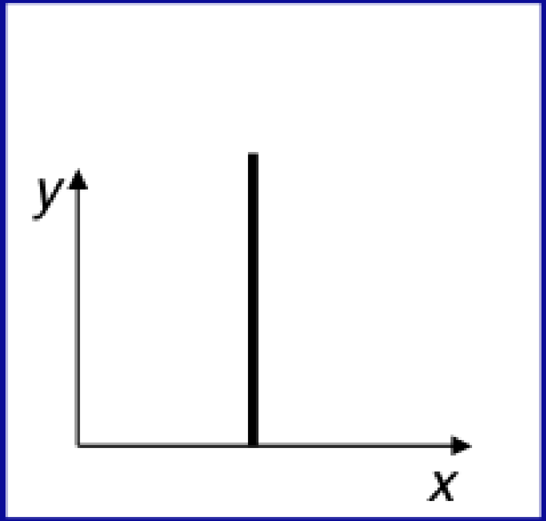
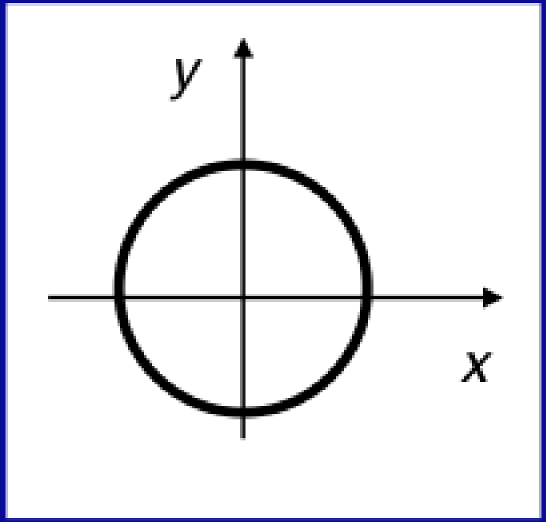


- **Implicit: f(x,y) = 0**

$$x^2 + y^2 - r^2 = 0$$

  +Easy to test if on the curve

  –Hard to generate points



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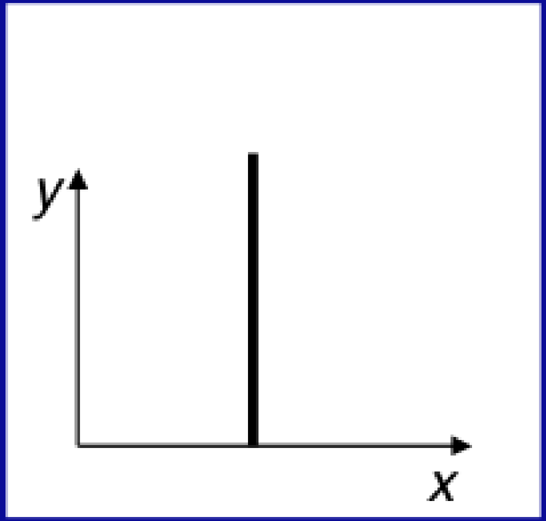

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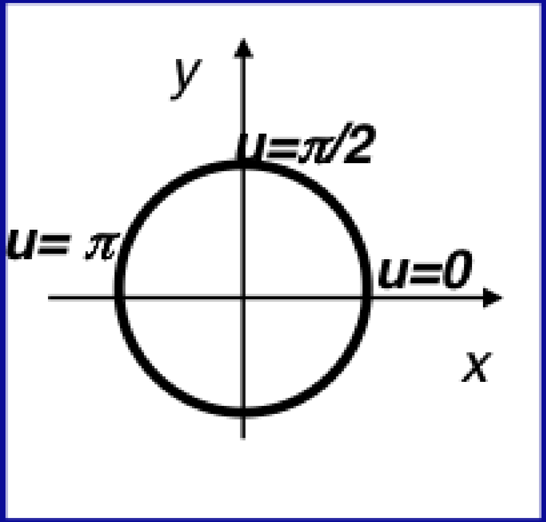
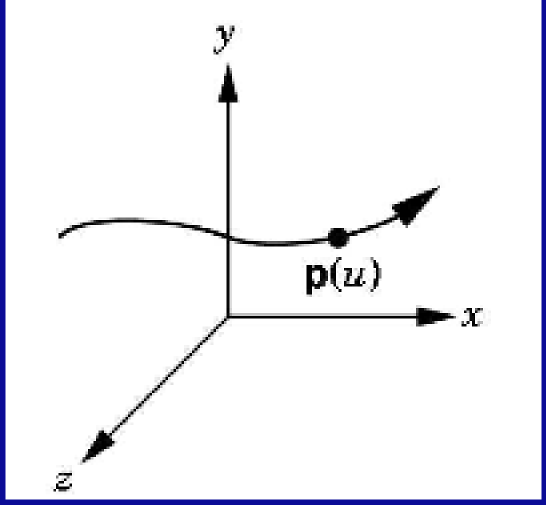


- **Parametric: (x,y) = (f(u), g(u))**

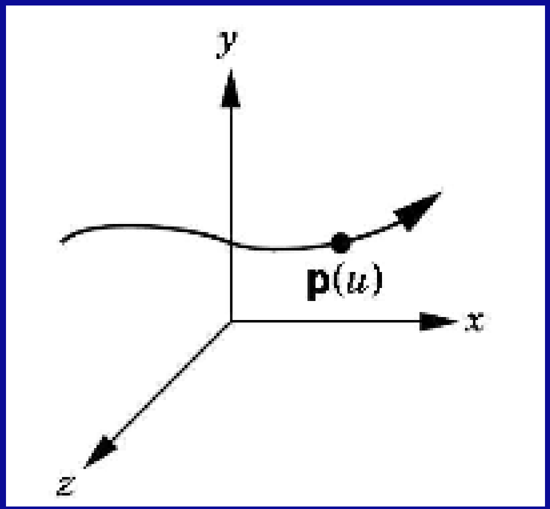
$$(x, y) = (\cos u, \sin u)$$

+Easy to generate points



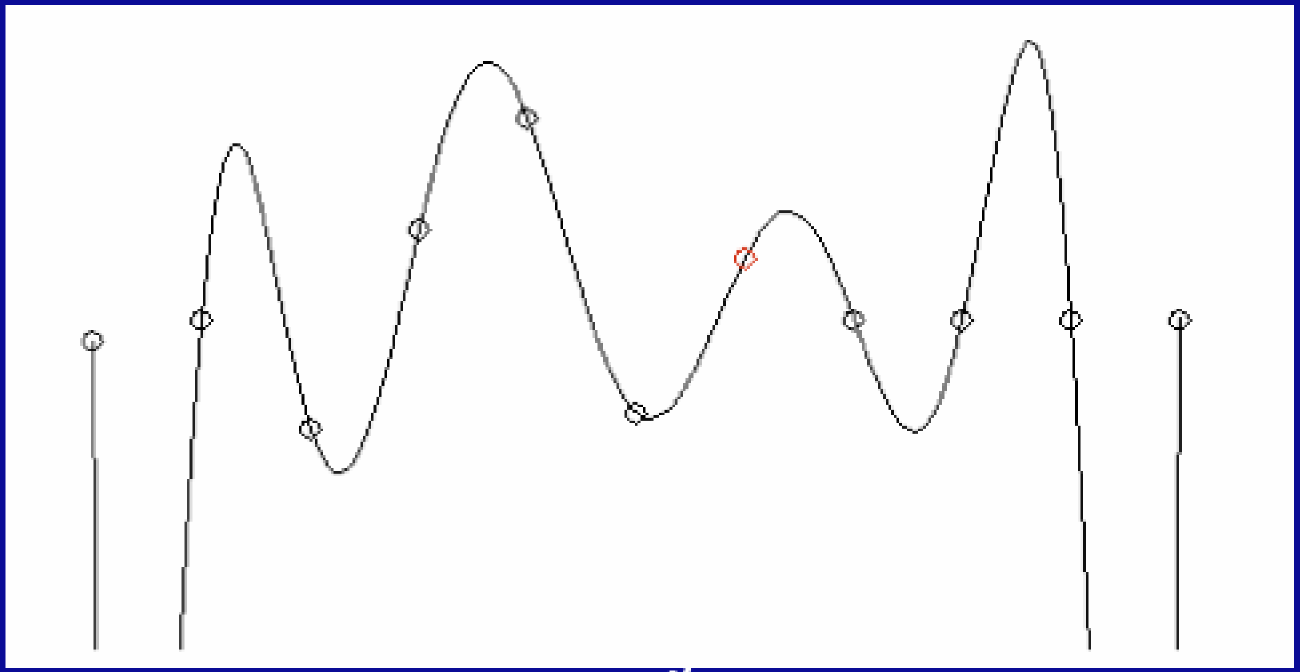
Parameterization of a Curve

- *Parameterization* of a curve: how a change in u moves you along a given curve in xyz space.



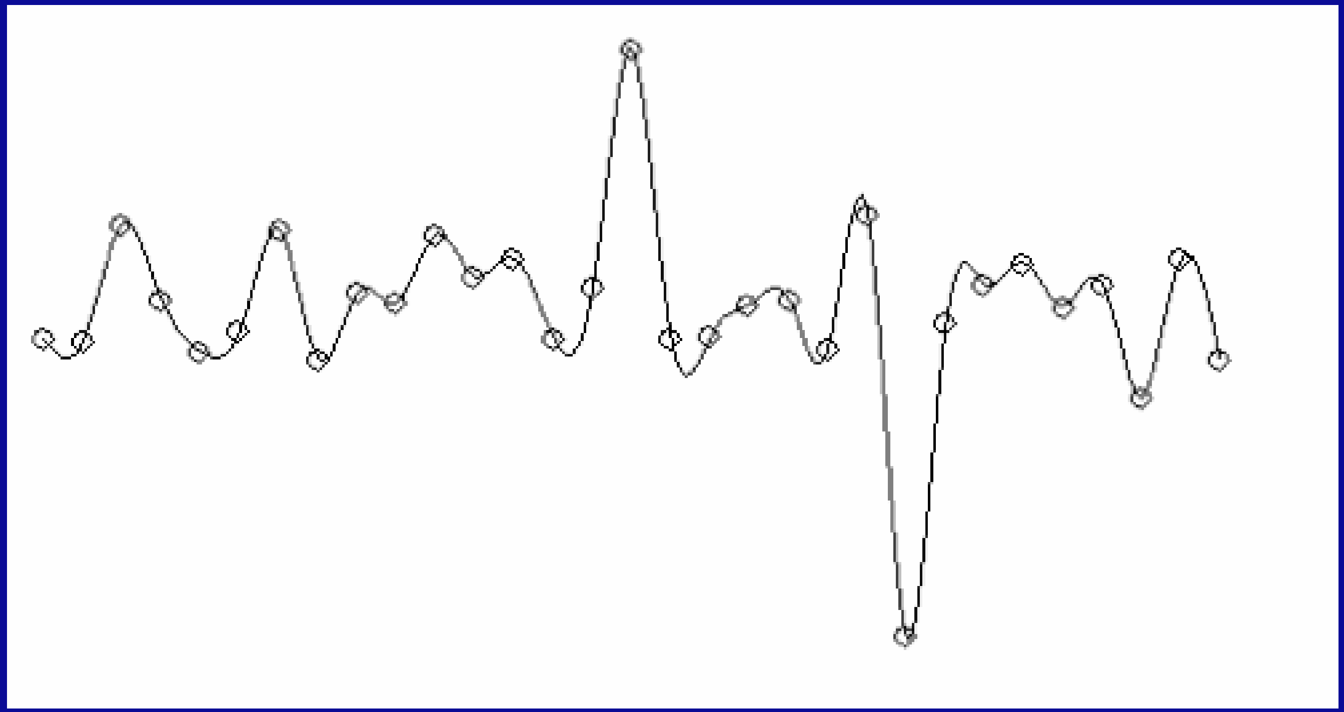
Polynomial Interpolation

- **An *n*-th degree polynomial fits a curve to n+1 points**
  - **called Lagrange Interpolation**
  - **result is a curve that is too wiggly, change to any control point affects entire curve (nonlocal) – *this method is poor***
- **We usually want the curve to be as smooth as possible**
  - **minimize the wiggles**
  - **high-degree polynomials are bad**



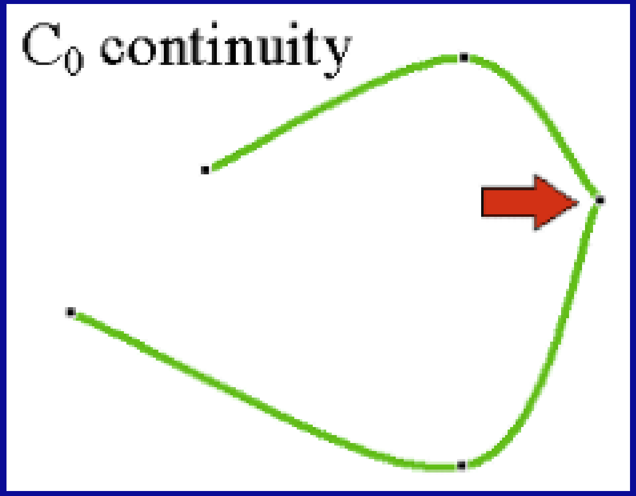
Splines: Piecewise Polynomials

- A spline is a *piecewise polynomial* - many low degree polynomials are used to interpolate (pass through) the control points
- *Cubic piecewise* polynomials are the most common:
  - piecewise definition gives local control

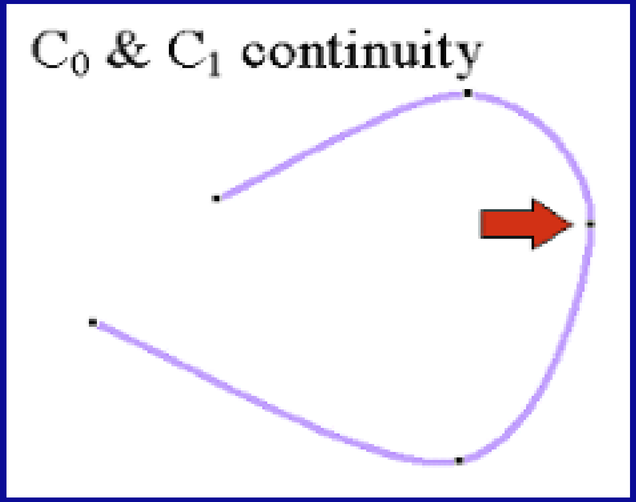


Piecewise Polynomials

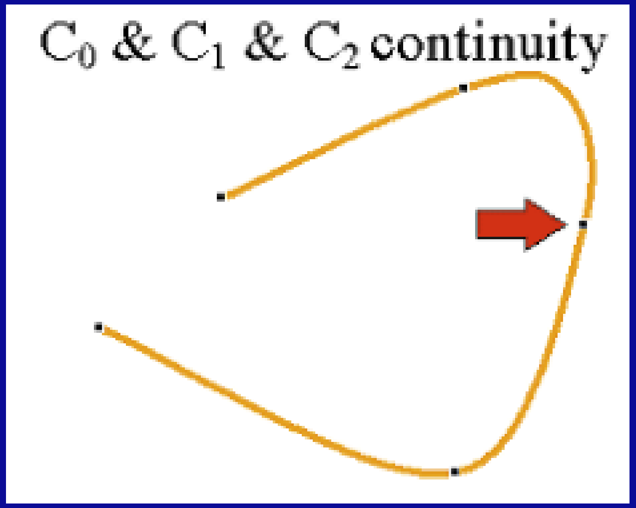
- Spline: lots of little polynomials pieced together
- Want to make sure they fit together nicely



Continuous in position



Continuous in position and tangent vector



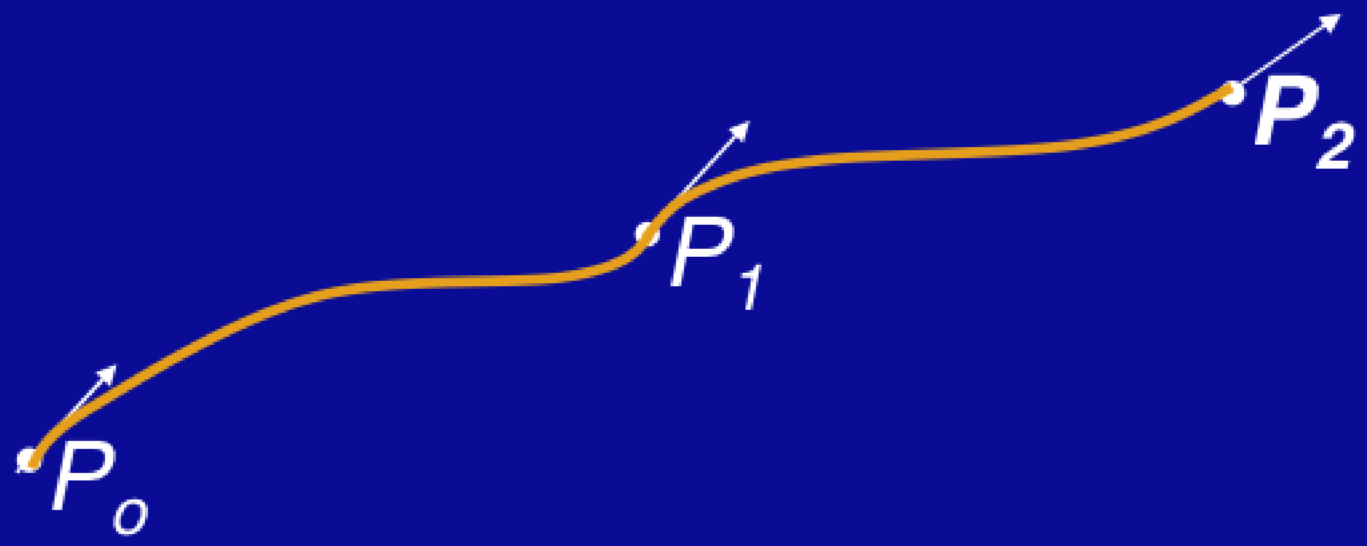
Continuous in position, tangent, and curvature

Splines

- Types of splines:
  - Hermite Splines
  - Catmull-Rom Splines
  - Bezier Splines
  - Natural Cubic Splines
  - B-Splines
  - NURBS

Hermite Curves

- **Cubic Hermite Splines**



That is, we want a way to specify the end points and the slope at the end points!

Splines

chalkboard

The Cubic Hermite Spline Equation

- Using some algebra, we obtain:

$$p(u) = \begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ \nabla p_1 \\ \nabla p_2 \end{bmatrix}$$

point that gets drawn

basis

control matrix (what the user gets to pick)

- This form typical for splines
  - basis matrix and meaning of control matrix change with the spline type

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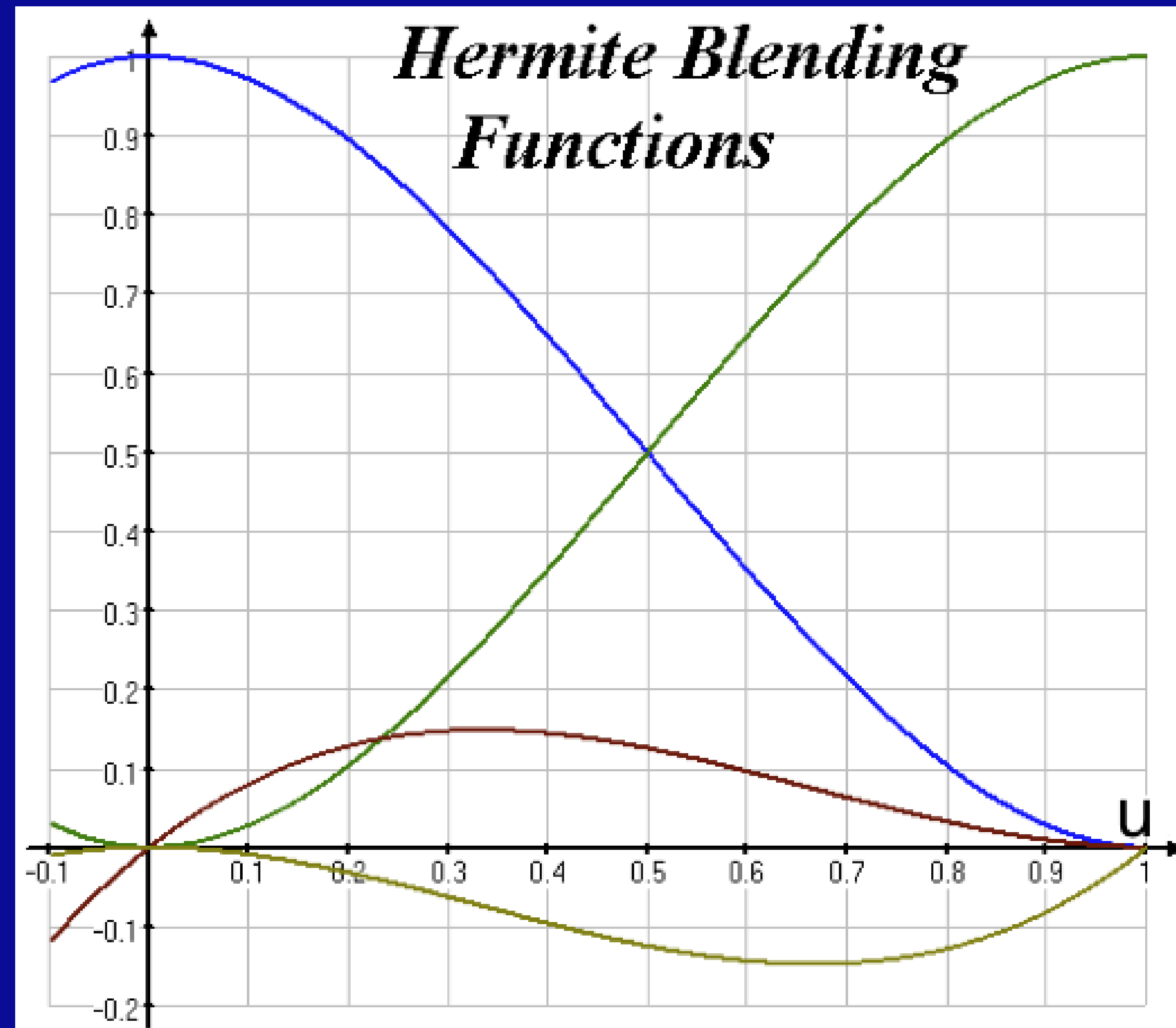
$$p(u) = \begin{bmatrix} 2u^3 - 3u^2 + 1 \\ -2u^3 + 3u^2 \\ u^3 - 2u^2 + u \\ u^3 - u^2 \end{bmatrix}^T \begin{bmatrix} p_1 \\ p_2 \\ \nabla p_1 \\ \nabla p_2 \end{bmatrix}$$

4 Basis Functions

Four Basis Functions for Hermite splines

$$p(u) = \begin{bmatrix} 2u^3 - 3u^2 + 1 \\ -2u^3 + 3u^2 \\ u^3 - 2u^2 + u \\ u^3 - u^2 \end{bmatrix}^T \begin{bmatrix} p_1 \\ p_2 \\ \nabla p_1 \\ \nabla p_2 \end{bmatrix}$$

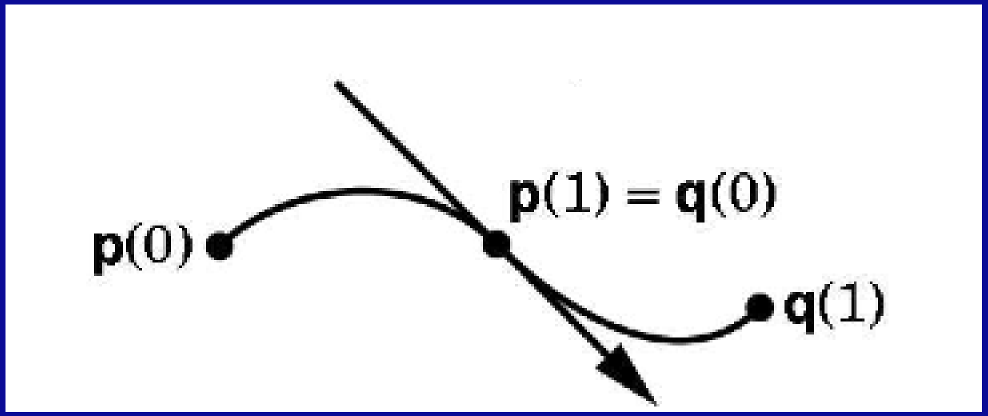
4 Basis Functions



Every cubic Hermite spline is a linear combination (blend) of these 4 functions

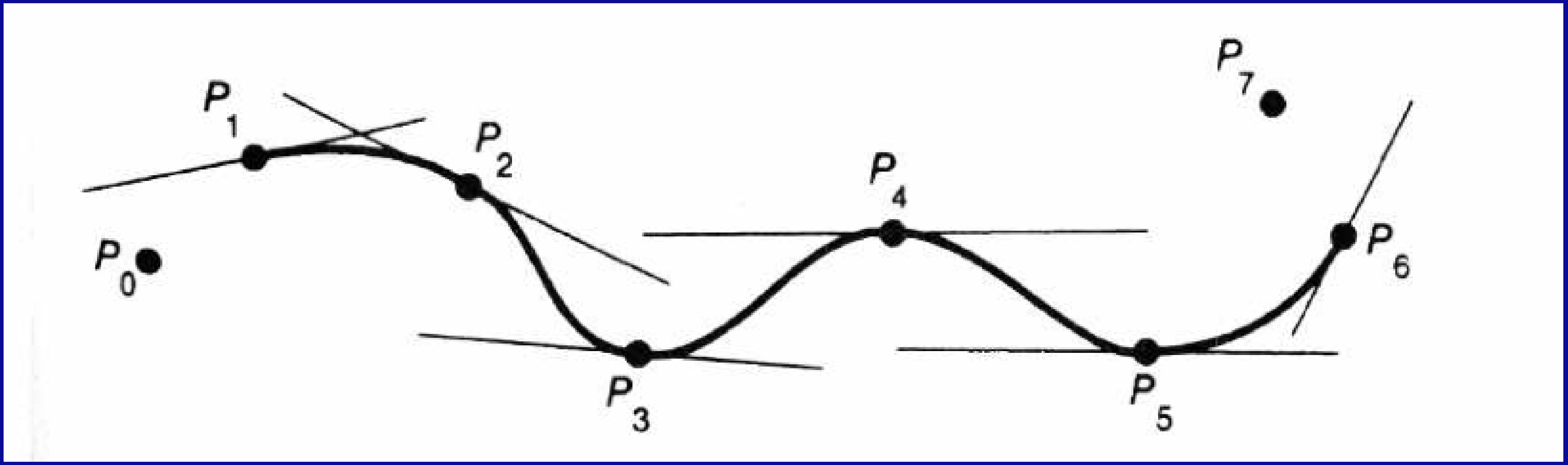
Piecing together Hermite Curves

- It's easy to make a multi-segment Hermite spline
  - each piece is specified by a cubic Hermite curve
  - just specify the position and tangent at each "joint"
  - the pieces fit together with matched positions and first derivatives
  - gives C1 continuity
- The points that the curve has to pass through are called *knots* or *knot points*



Catmull-Rom Splines

- With Hermite splines, the designer must specify all the tangent vectors
- Catmull-Rom: an interpolating cubic spline with *built-in C^1 continuity*.



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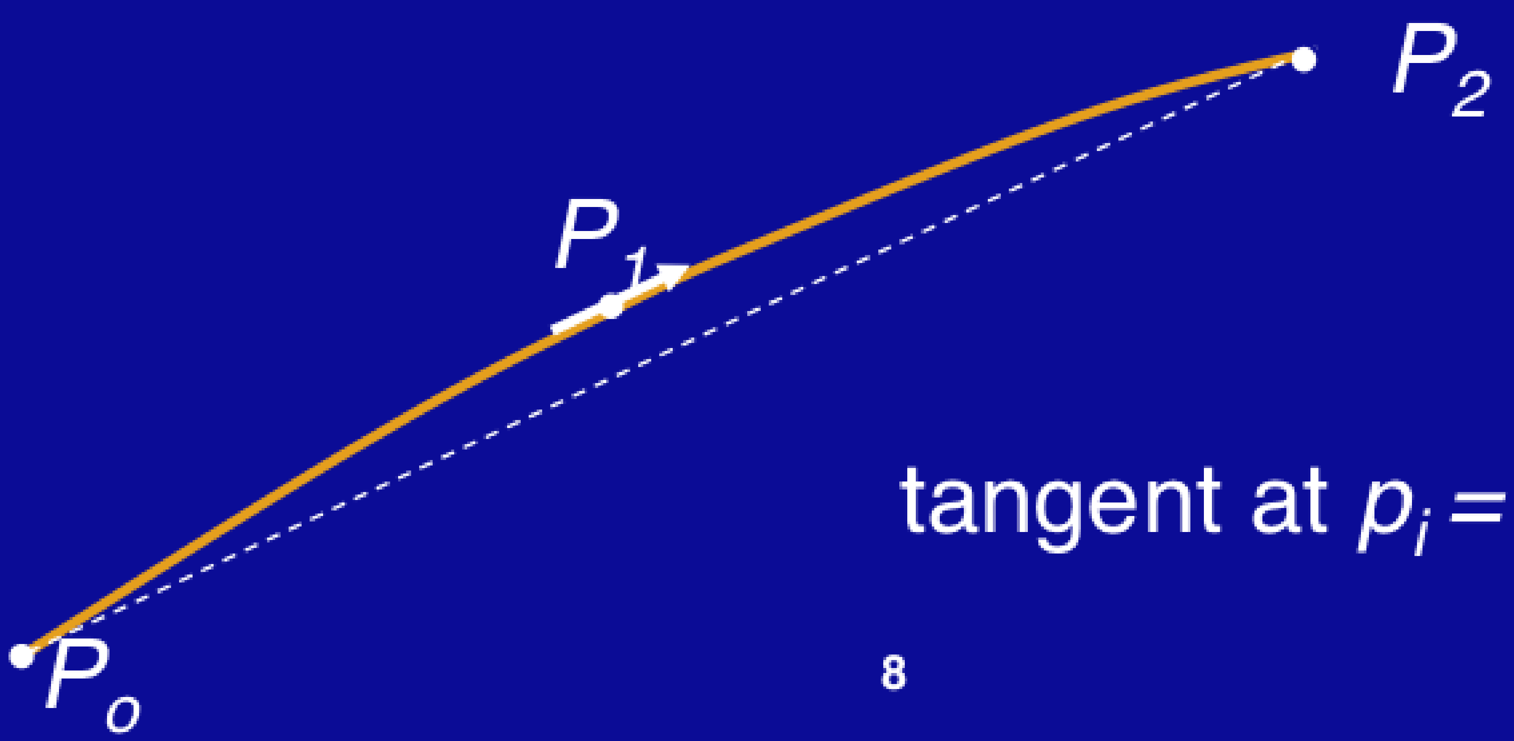

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tangent at $p_i = s(p_{i+1} - p_{i-1})$

Catmull-Rom Spline Matrix

$$p(u) = \begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix} \begin{bmatrix} -s & 2-s & s-2 & s \\ 2s & s-3 & 3-2s & -s \\ -s & 0 & s & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix}$$

spline coefficients **CR basis** **control vector**

- **Derived similarly to Hermite**
- **Parameter s is typically set to s=1/2.**

Cubic Curves in 3D

- **Three cubic polynomials, one for each coordinate**
  - $x(u) = a_x u^3 + b_x u^2 + c_x u + d_x$
  - $y(u) = a_y u^3 + b_y u^2 + c_y u + d_y$
  - $z(u) = a_z u^3 + b_z u^2 + c_z u + d_z$
- **In matrix notation**

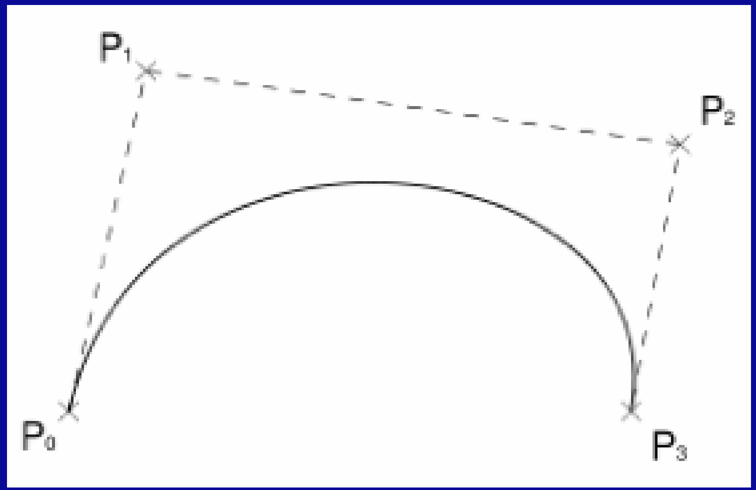
$$\begin{bmatrix} x(u) & y(u) & z(u) \end{bmatrix} = \begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix} \begin{bmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \\ d_x & d_y & d_z \end{bmatrix}$$

Catmull-Rom Spline Matrix in 3D

$$\begin{bmatrix} x(u) & y(u) & z(u) \end{bmatrix} = \begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix} \underbrace{\begin{bmatrix} -s & 2-s & s-2 & s \\ 2s & s-3 & 3-2s & -s \\ -s & 0 & s & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}}_{\textbf{CR basis}} \underbrace{\begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \\ x_4 & y_4 & z_4 \end{bmatrix}}_{\textbf{control vector}}$$

Bezier Curves*

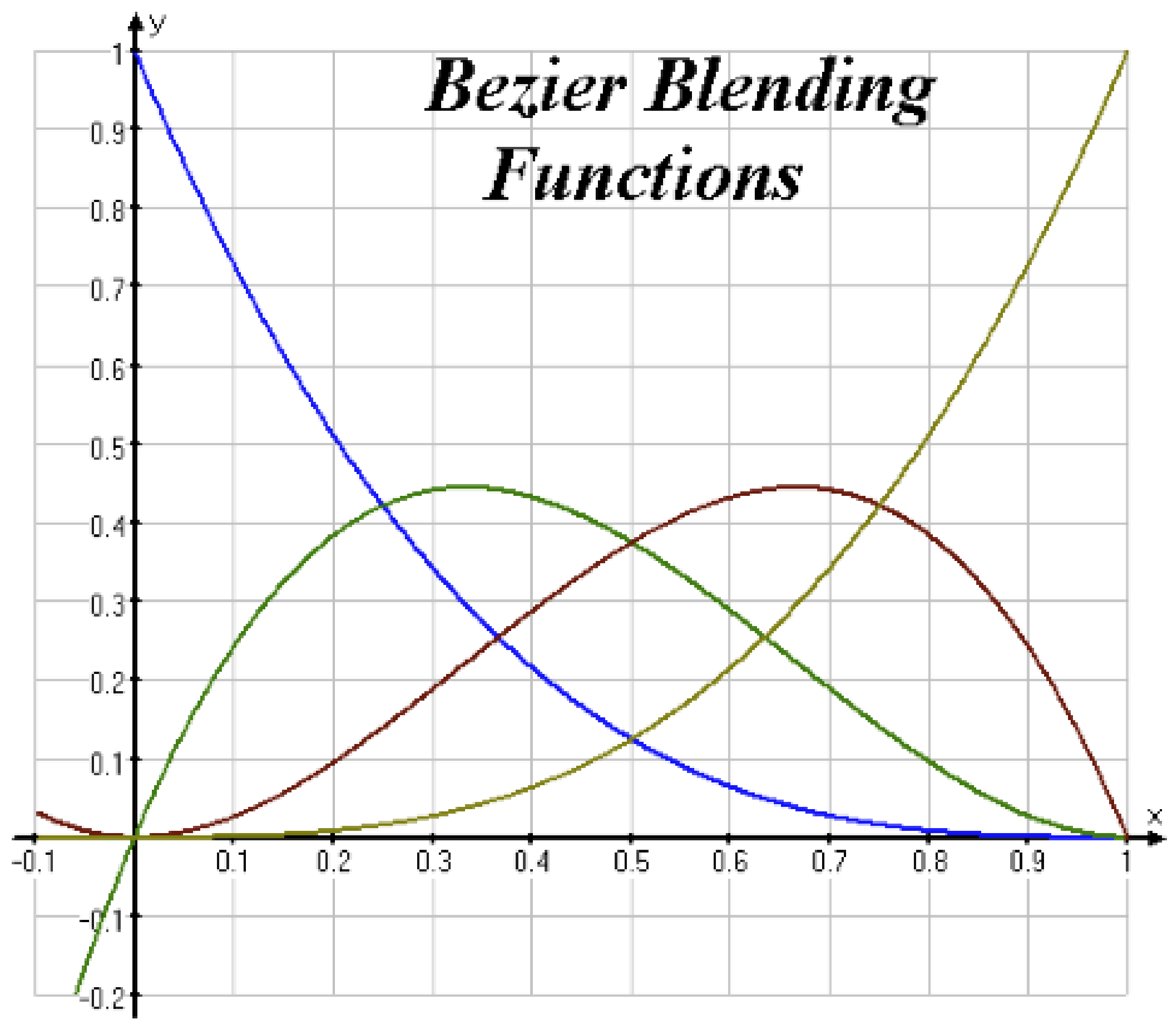
- Another variant of the same game
- Instead of endpoints and tangents, four control points
  - points P0 and P3 are on the curve: P(u=0) = P0, P(u=1) = P3
  - points P1 and P2 are off the curve
  - P'(u=0) = 3(P1-P0), P'(u=1) = 3(P3 – P2)
- Convex Hull property
  - curve contained within convex hull of control points
- Gives more control knobs (series of points) than Hermite
- Scale factor (3) is chosen to make "velocity" approximately constant



The Bezier Spline Matrix*

$$\begin{bmatrix} x & y & z \end{bmatrix} = \begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix} \underbrace{\begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}}_{\textbf{Bezier basis}} \underbrace{\begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \\ x_4 & y_4 & z_4 \end{bmatrix}}_{\substack{\textbf{Bezier} \\ \textbf{control vector}}}$$

Bezier Blending Functions*



$$p(t) = \begin{bmatrix} (1-t)^3 \\ 3t(1-t)^2 \\ 3t^2(1-t) \\ t^3 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix}$$

Also known as the order 4, degree 3 Bernstein polynomials

Nonnegative, sum to 1

The entire curve lies inside the polyhedron bounded by the control points

Splines with More Continuity?

- How could we get C^2 continuity at control points?

- Possible answers:
  - Use higher degree polynomials

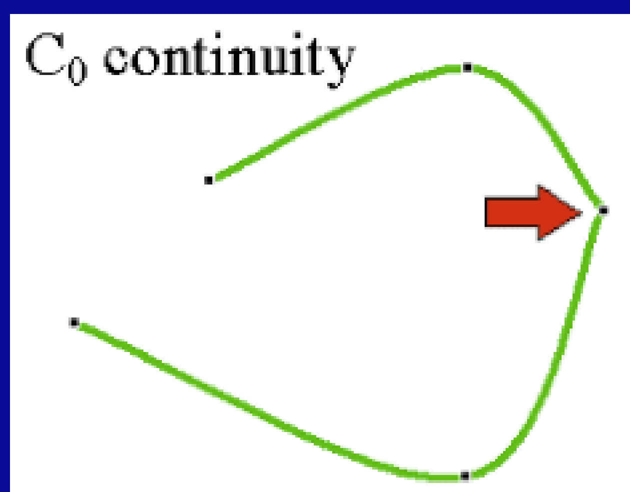
    degree 4 = quartic, degree 5 = quintic, … but these get computationally expensive, and sometimes wiggly
  - Give up local control   natural cubic splines

    A change to any control point affects the entire curve
  - Give up interpolation   cubic B-splines

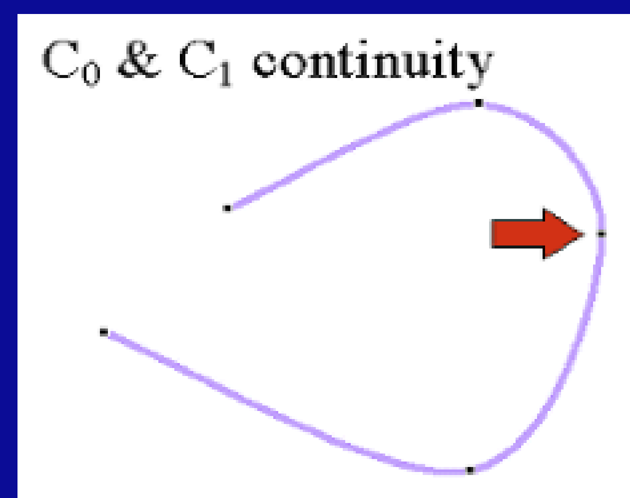
    Curve goes near, but not through, the control points

Piecewise Polynomials

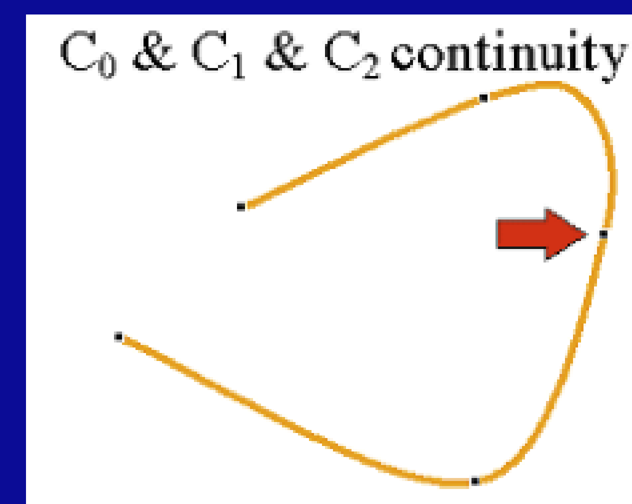
- **Spline: lots of little polynomials pieced together**
- **Want to make sure they fit together nicely**



Continuous in position



Continuous in position and tangent vector



Continuous in position, tangent, and curvature

Comparison of Basic Cubic Splines

Type	Local Control	Continuity	Interpolation
Hermite	YES	C1	YES
Bezier	YES	C1	YES
Catmull-Rom	YES	C1	YES
Natural	NO	C2	YES
B-Splines	YES	C2	NO

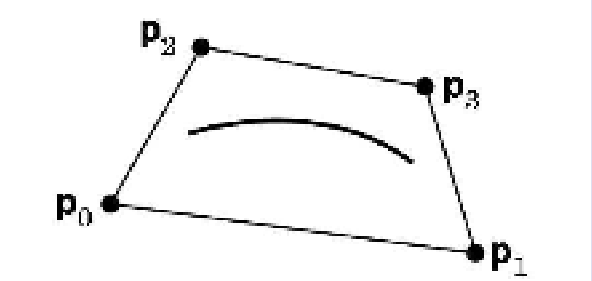
- Summary
  - Can’t get C2, interpolation and local control with cubics

Natural Cubic Splines*

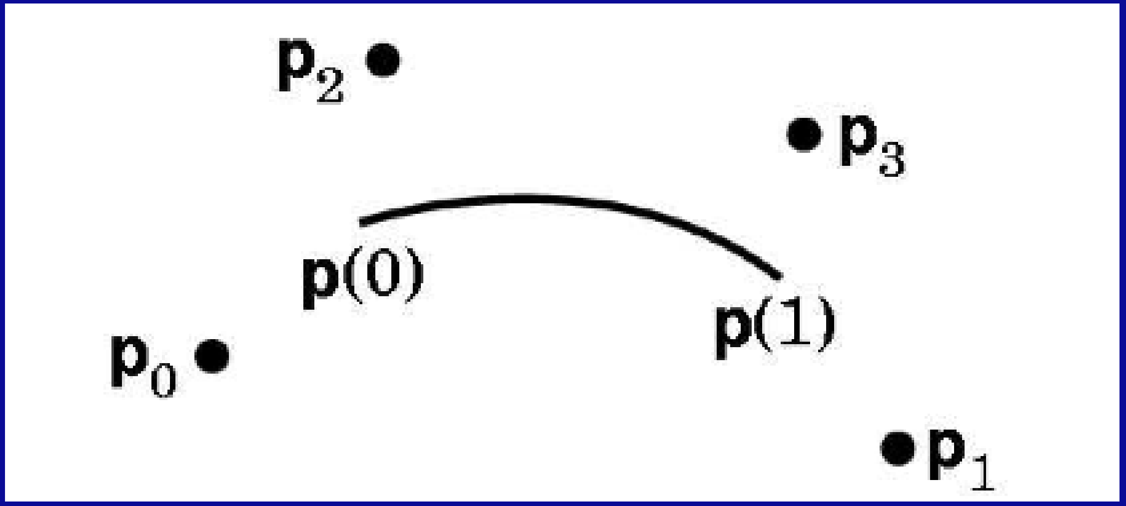
- If you want 2nd derivatives at joints to match up, the resulting curves are called *natural cubic splines*
- It's a simple computation to solve for the cubics' coefficients. (See *Numerical Recipes in C* book for code.)
- Finding all the right weights is a *global* calculation (solve tridiagonal linear system)

B-Splines*

- **Give up interpolation**
  - the curve passes *near* the control points
  - best generated with interactive placement (because it's hard to guess where the curve will go)



- **Curve obeys the convex hull property**
- **C2 continuity and local control are good compensation for loss of interpolation**

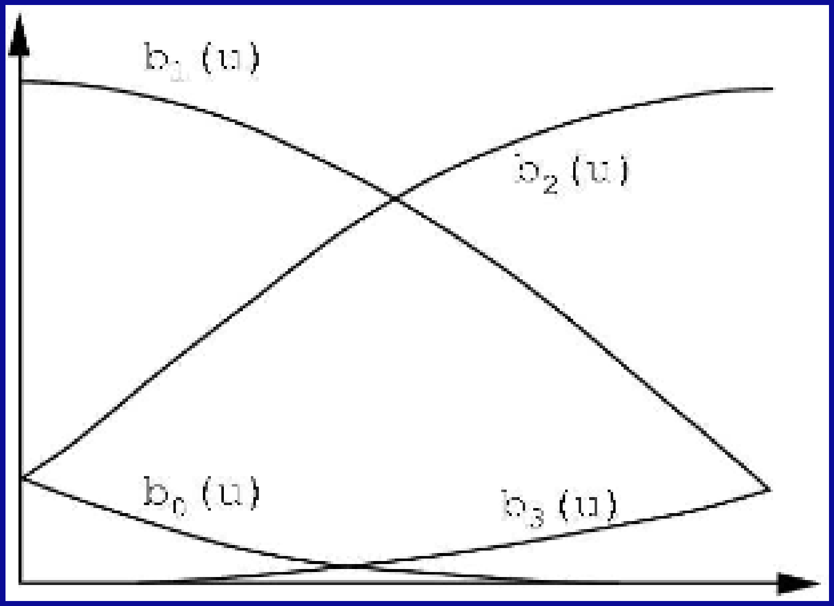


B-Spline Basis*

- We always need 3 more control points than spline pieces

$$M_{Bs} = \frac{1}{6}\begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{bmatrix}$$

$$G_{Bsi} = \begin{bmatrix} P_{i-3} \\ P_{i-2} \\ P_{i-1} \\ P_i \end{bmatrix}$$



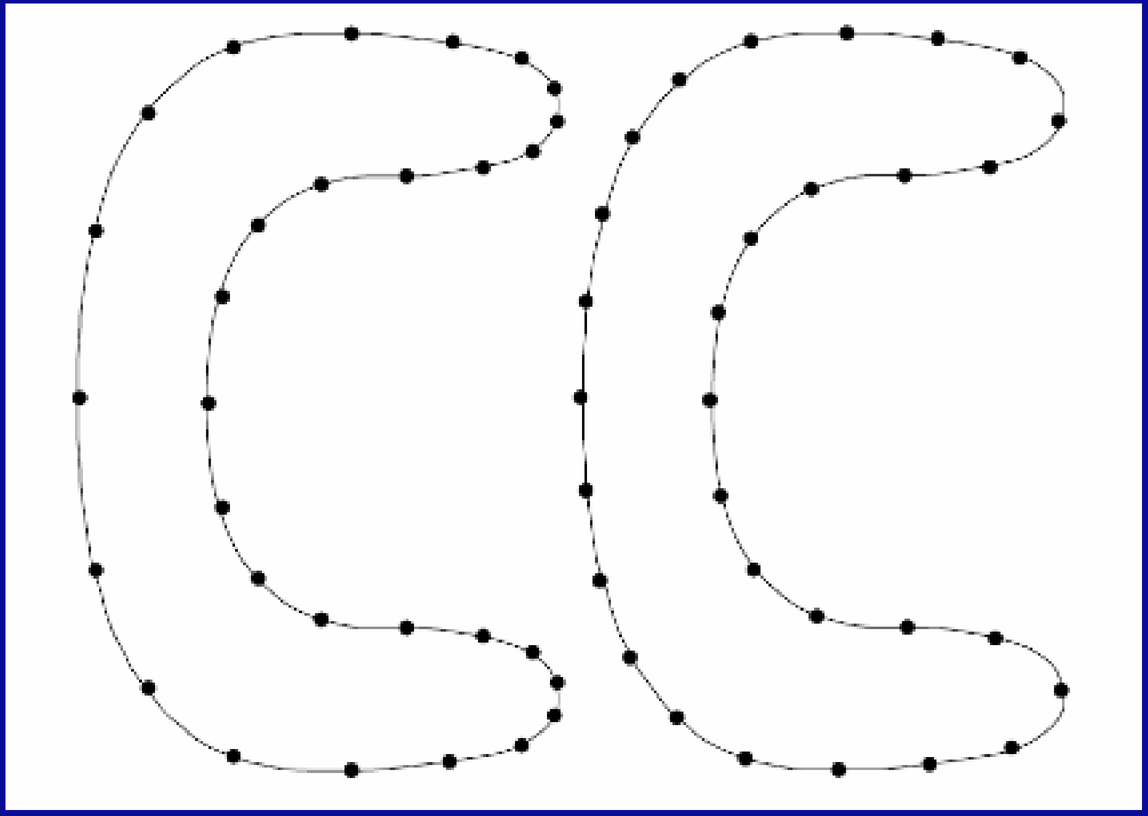
How to Draw Spline Curves

- **Basis matrix eqn. allows same code to draw any spline type**
- **<u>Method 1</u>: brute force**
  - **Calculate the coefficients**
  - **For each cubic segment, vary u from 0 to 1 (fixed step size)**
  - **Plug in u value, matrix multiply to compute position on curve**
  - **Draw line segment from last position to current position**

$$\begin{bmatrix} x & y & z \end{bmatrix} = \begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix} \underbrace{\begin{bmatrix} -s & 2-s & s-2 & s \\ 2s & s-3 & 3-2s & -s \\ -s & 0 & s & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}}_{\text{CR basis}} \underbrace{\begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \\ x_4 & y_4 & z_4 \end{bmatrix}}_{\text{control vector}}$$

How to Draw Spline Curves

- What's wrong with this approach?
  - –Draws in even steps of u
  - –Even steps of u $\neq$ even steps of x
  - –Line length will vary over the curve
  - –Want to bound line length
    - »too long: curve looks jagged
    - »too short: curve is slow to draw

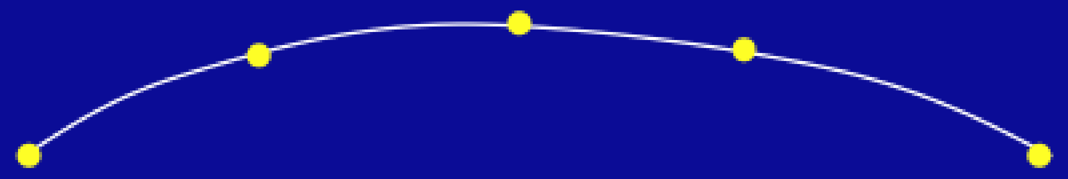


Drawing Splines, 2

- **Method 2: recursive subdivision - vary step size to draw short lines**

```
Subdivide(u0,u1,maxlinelength)
    umid = (u0 + u1)/2
    x0 = P(u0)
    x1 = P(u1)
    if |x1 - x0| > maxlinelength
       Subdivide(u0,umid,maxlinelength)
       Subdivide(umid,u1,maxlinelength)
    else drawline(x0,x1)
```

- **Variant on Method 2 - subdivide based on curvature**
  - **replace condition in "if" statement with straightness criterion**
  - **draws fewer lines in flatter regions of the curve**



In Summary...

- **Summary:**
  - piecewise cubic is generally sufficient
  - define conditions on the curves and their continuity

- **Things to know:**
  - basic curve properties (what are the conditions, controls, and properties for each spline type)
  - generic matrix formula for uniform cubic splines $x(u) = uBG$
  - given definition derive a basis matrix