15-456 Computational Geometry, Spring 2013

Homework 4 Due: 22 March 2013

Guidelines: Please justify all answers in a succienct (yet complete) manner. In particular, when presenting an algorithm the code if any should be presented at a high level. A full algorithm will contain the input, the output, and any loop invariants.

Question	Points	Score
1	30	
2	15	
3	30	
4	30	
5	20	
Total:	125	

(30) 1. **Staircases**

Let P be a set of n points in the plane. A point $p \in P$ is Pareto-optimal if no other point in P is both above and to the right of p. The sorted sequence of Pareto-optimal points describes a staircase with all the points in P below and to the left. The staircase layers of P are defined recursively as follows. The first staircase layer is just the staircase; for all $k > 1$, the kth staircase layer is the staircase of P after the points in the first $k - 1$ staircase layers have been deleted.

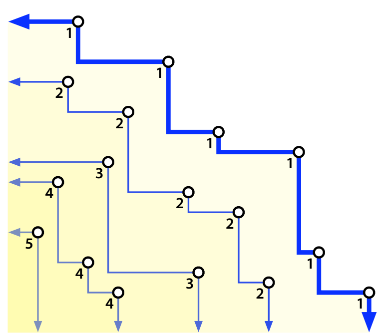

Figure 1: set of points with five staircase layers.

1. Describe and analyze an algorithm to compute the staircase layers of P in $O(n \log n)$ time. Your algorithm should label each point with an integer indicating which staircase layer contains it, as shown in the Figure 1 above.

2. Describe and analyze an algorithm to compute the staircase of P in $O(n \log p)$ time, where p is the number of Pareto-optimal points. [Hint: There are at least two different ways to do this.]

(15) 2. **Lifting Edge Flips** Consider a convex quadrilateral $abcd$ in $\mathbb{R}^2$. The two diagonals of $abcd$ are ac and bd. WLOG, assume that the triangulation of the four points obtained by the quadrilateral and the edge ac is the Delaunay triangulation of the four points. If we lift the points a, b, c, and d to the paraboloid $(x, y, x^2 + y^2)$, then the four points (since not cocircular) are the vertices of a non-degenerate tetrehedron in $\mathbb{R}^3$.

The two edges ac and bd intersect in the plane at a point p, but the corresponding edges connecting the lifted points do not intersect. Above the point p, which line is lower? Be sure to explain your reasoning.

(30) 3. **Farthest Point Voronoi Diagram** ($30 = 5 + 10 + 10 + 5$) Given a point set P of n points, the farthest point Voronoi diagram $V_{n-1}(P)$ partitions the domain into regions with the common farthest point(s) in the set P (as opposed to the same closest point as is the case in the Voronoi diagram). This decomposition has faces (2-dimensional regions with a unique farthest point), edges (1-dimentional regions with exactly two farthest points in the set P), and vertices (a discrete set of points with more than two farthest points).

(a) Draw an example of the farthest point Voronoi diagram with 9 sites, 5 of which are on the convex hull.

(b) Prove that the faces of $V_{n-1}(P)$ are convex.

(c) Prove that the only sites that have faces in $V_{n-1}(P)$ are on the boundary of the convex closure of P.

(d) Give an $O(n \log n)$ algorithm to compute the farthest point Voronoi diagram.

(30) 4. **Simple Paths and Convex Hull**

Suppose that $P = \{p_1, \ldots, p_n\}$ is a set of points in the plane. We say the the sequences of distinct points $Path = (p_1, \ldots, p_k)$ is a simple path if the line segments $l_i = [p_i p_{i+1}]$ are disjoint except for $l_i \cap l_{i+1} = p_{i+1}$. We may also allow $p_1 = p_k$ and in this case $l_{k-1} \cap l_1 = p_k$.

In the following questions we shall investigate the relation between finding a simple path of a set of points and finding their convex hull.

1. Design an algorithm for finding a simple path through all points in P. Make your algorithm as time efficient as possible.

2. In class we showed that computing the convex hull of n points in a comparison basedN - model requires $\Omega(n \log n)$ time. Show that given a simple path for these points one can find the convex hull in $O(n)$ time.

   HINT:

The idea is to run a variant of incremental convex hull where we add the points in the order they appear on the path. Suppose we are give a simple path $Path = (p_1, \ldots, p_n)$ on n distinct points and for simplicity no three are collinear. We start by constructing the triangle from the first three points and storing it as a doubly linked list of edges and recording which vertex is connected to the remain points on the path.

Let $I = \{i \mid p_i \in CH(p_1, \ldots, p_i)\}$ We will for each $i \in I$ incrementally compute the convex hull of $(p_1, \ldots, p_i)$. Make sure your algorithm handles the case when the point p_{i+1} is interior to $CH(p_1, \ldots, p_i)$.

Use amortized analysis to show that your algorithm runs in $O(n)$ time.

3. Show that in general any comparison based algorithm that finds a simple path of the points in P requires $\Omega(n \log n)$ comparisons.

(20) 5. **Compute clustering radius** ($20 = 10 + 10$) Let C and P be two given sets of points in the plane, such that $k = |C|$ and $n = |P|$.

Let $r = \max_{p \in P} \min_{c \in C} ||c - p||$ be the **covering radius** of P by C (i.e., if we place a disk of radius r around each point of C, all those disks cover the points of P). The goal of this problem is to approximate r which we shall do in stages.

(a) Give an $O(n + k)$ time algorithm which computes the following predicate:

$$Test(\alpha) = \begin{cases} \text{yes} & \text{if } r < \alpha \\ p \in P \text{ such that } ||p, C|| > r & \text{if } 2\sqrt{2}\alpha < r \\ \text{either case above} & \text{otherwise} \end{cases}$$

(b) Give an $O(n + k \log n)$ expected time algorithm that outputs a number α, such that $r \leq \alpha \leq 10r$.