15-456
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Point Location Prob

Trapezoidal Decomp

Def Planar Straight Line Graph (PSLG) $G=(V,E)$

Todays Design assumption: no 2 vertices with same x.

n-line seg $2n \geq$ vertices

Point location Prob

Input: $G=(V,E)$ PSLG

Preprocess:

Query: Given p find face containing p.

Simple 2D Alg

1) Sort V by x-value
2) a) Decomp G in slabs
   b) sort each slab
3) Query P:
   a) find slab containing P. $O(\log n)$
   b) find P in slab $O(\log n)$

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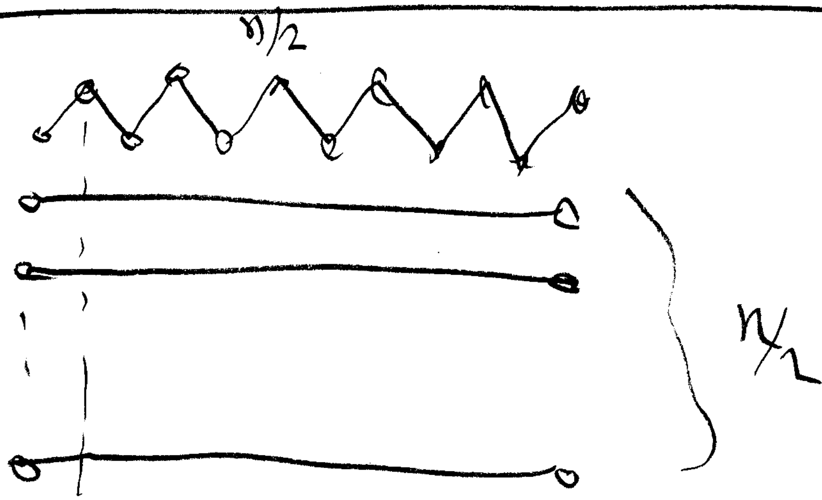
eg



$n/2$ slabs of size $n/2$!

$\Omega(n^2)$ answers!

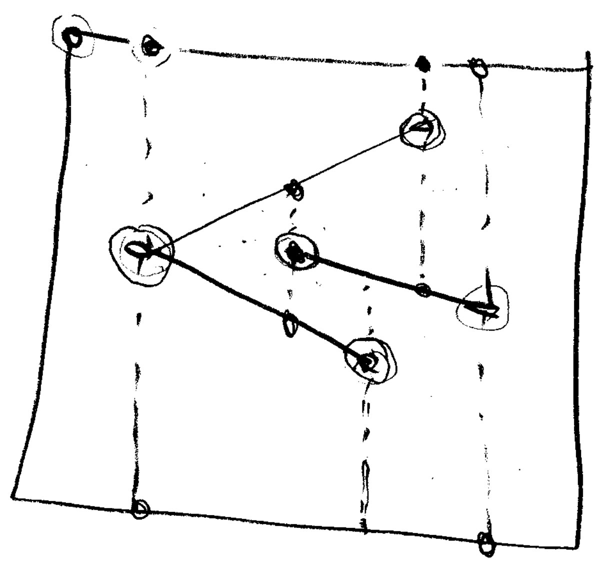
Trap Decomp

Input: n - line seg (no two endpoints with same x)

Output:

a) Add bounding box

b) extend up & down each endpoint until it hits a seg.



Lemma a) at most $6n+4$ vertices

b) at most $3n+1$ traps

a) 4 on bounding box

$2n$ input

$2(2n)$ intersections

b) each trap has unique left endpoint

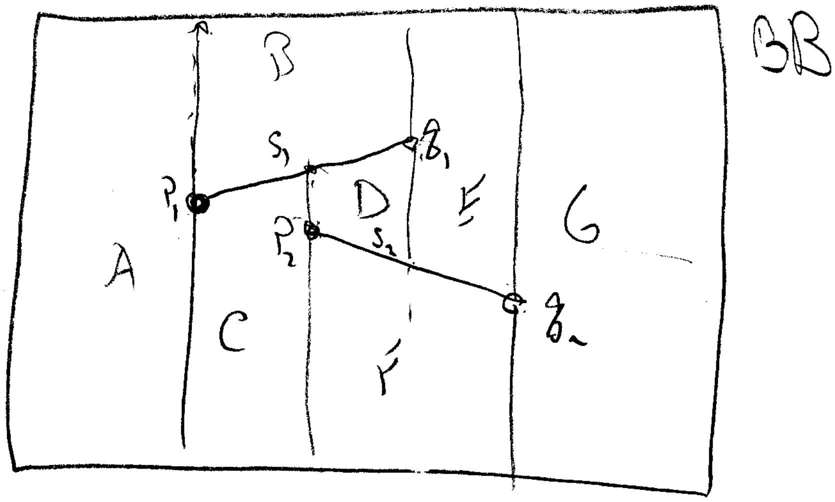
each right endpoint "sees" one trap

each left " " 2 traps

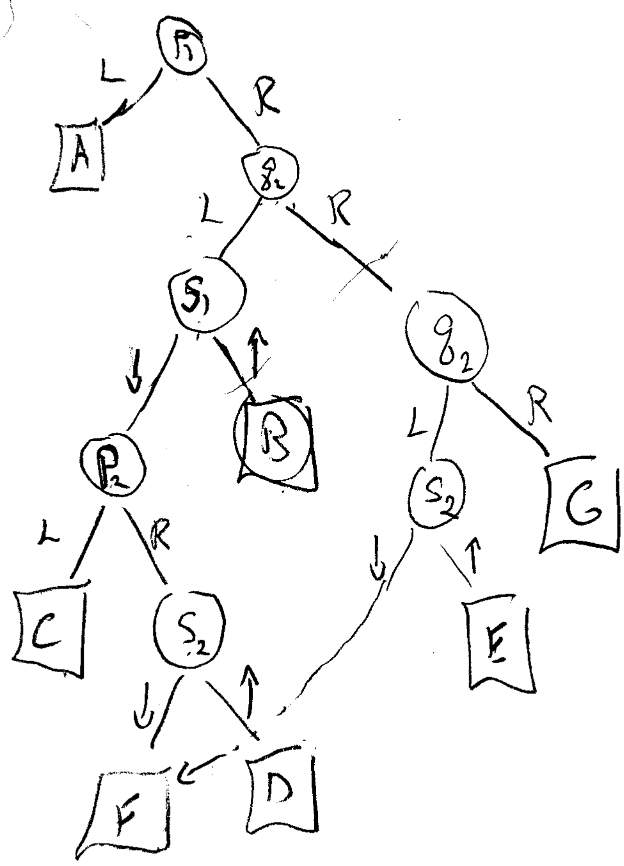
bb "sees" one trap

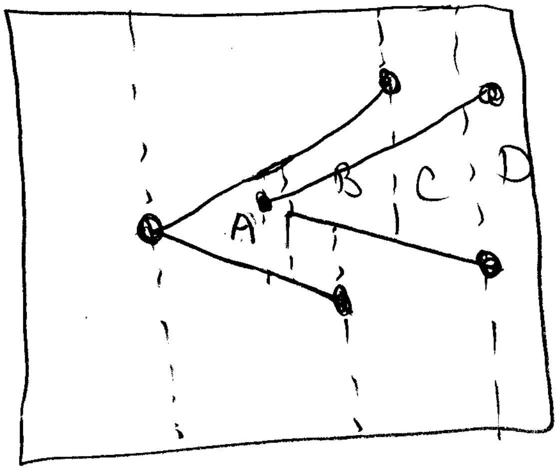
$3n+1$

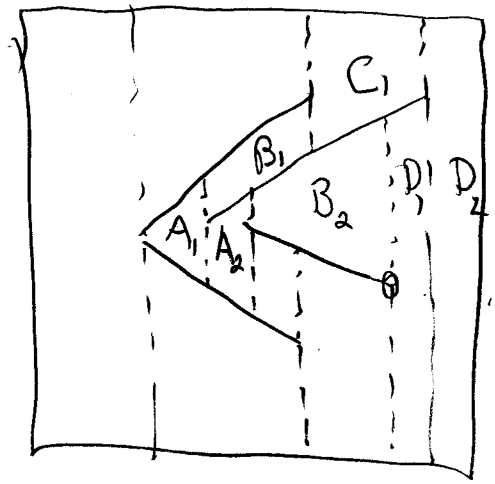
Point location in Trap decomp.

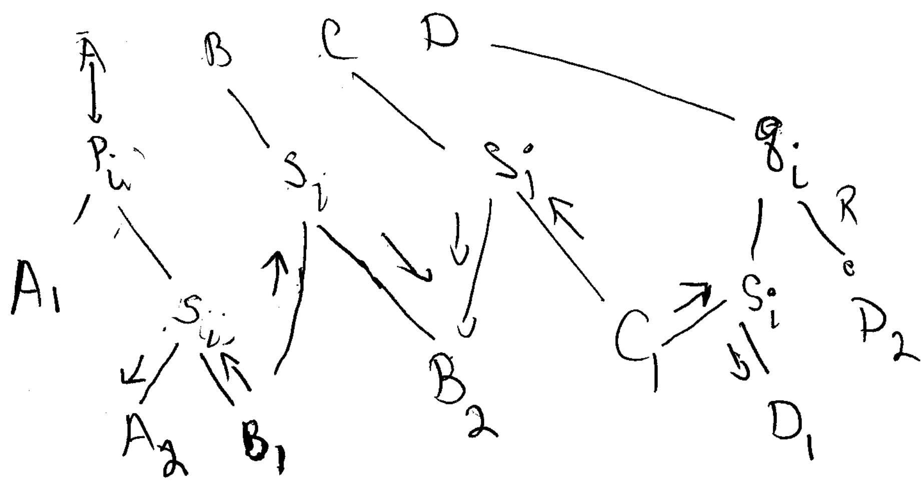


Assume order S_1, S_3




A
B
C
D


C1
B1
D1
D2
B2
A1
A2


A
B
C
D
pi
si
sj
qi
R
A1
si
B2
C1
si
D2
A2
B1
D1

Random Incremental

<u>Alg</u> Trap Map(S)

Input: Segments $S = s_1 \cdots s_n$

Output: 1) Trap Map τ

2) Point Location DS $\mathcal{D}$.

1) Init: Build bounding box B & $\mathcal{D}$ for B

2) Randomly order $s_1, \ldots, s_n$

3) for $i = 1$ to n.

a) Find trap containing left-endpoint of s_i

b) Stitch in s_i from left-to-right.

c) Update $\mathcal{D}$.

Following pointers

1) left-vertical(Δ)

2) right-vertical(Δ)

3) left-trap(edge)

4) right-trap(edge)

Thm Trap Map

1) $O(n \log n)$ expected time to build
2) $|\mathcal{D}| = O(n)$ expected
3) Search time is $O(\log n)$ expected.

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note 1 Expectation is over $n!$ orderings of S.

note 2 Trap of $S_i \subseteq S$, $\mathcal{T}(S_i)$ is indep of order of S_i.

note 3 Each update of $\mathcal{D}$ increases depth ≤ 3,

Fix $q \in \mathcal{B}$, bb.

Consider random variable:

$$X = \text{depth}(q, \mathcal{D})$$

Goal: Bound $E(X)$

Consider X_i = # levels added when adding ith edge.

note $X = \sum X_i$

Thus $E(X) = \sum E(X_i)$.

<u>note</u> $X_i \leq 3$

Consider fixed subset $S_i \subseteq S$ $|S_i| = i$.

Let $\Delta_q(S_i) \equiv$ Trap containing q in $\mathcal{T}(S_i)$.

$$P_i = \text{Prob}\left[\Delta_q(S_i) \neq \Delta_q(S_{i-1})\right]$$

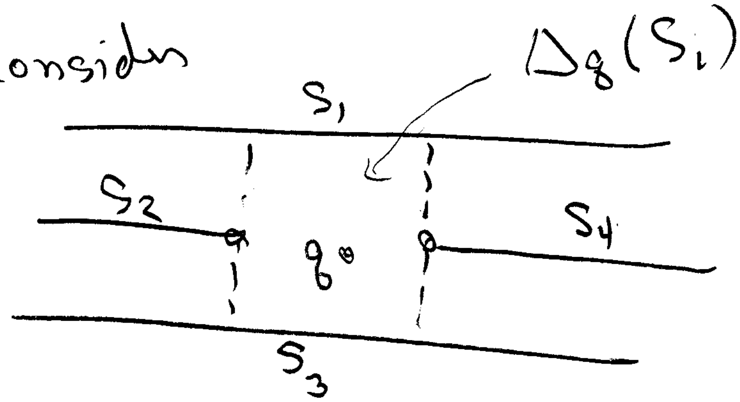
Thus $E(X_i) \leq 3P_i$

<u>Claim</u> $P_i \leq 4/i$

Use Backwards Analysis!

Of i seg remove one at random

Consider



Only removing $s_1 \cdots s_4$ changes $\Delta_q(S_i)$

$E(X_i) \leq \frac{3 \cdot 4}{i}$ (note: indep of set S_i)

$\therefore$ $E(X) \leq 12 H_n$ the nth harmonic number!

2) The size of $\mathcal{D}$

node = # Traps + internal nodes.

traps = $O(n)$ (Thm)

K_i = # new traps at time i.

$K_i - 1$ = # internal nodes

Backwards analysis

K_i = # traps removed when removing random seg

traps = $3i+1$

$p_i \leq 4/i$

$$E(K_i) \leq (3i+1)\left(\frac{4}{i}\right) = O(1)$$

$$E(|D|) = \sum E(K_i) = O(n)$$

1) Build Time

a) Expect time to find left endpoint

$O(\log i)$

b) Expect work to stitch in seg is

$O(1)$.

Total expect work $O(\sum_{i=1}^{n} \log i) = O(n \log n)$