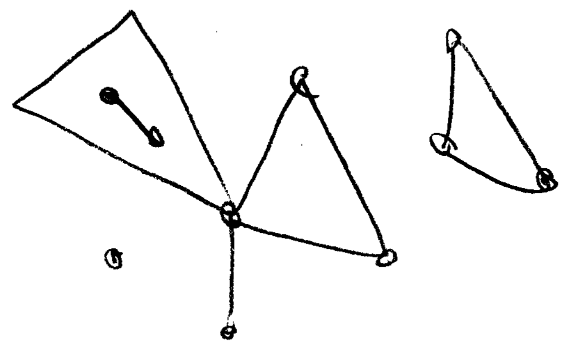
Representing Topological Information

15-456
2/20/13

Planar Subdivision ≡ Partition of the plane into

Vertices
Edges
Faces



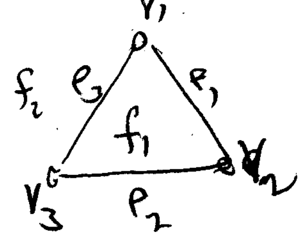
Vertices are closed
Edges & Faces are open
Face ≡ maximally connected region

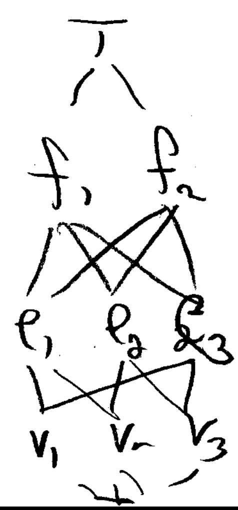
Degeneracies not allowed
1) Isolated vertex on a Face or Edges

Types of Topological Info

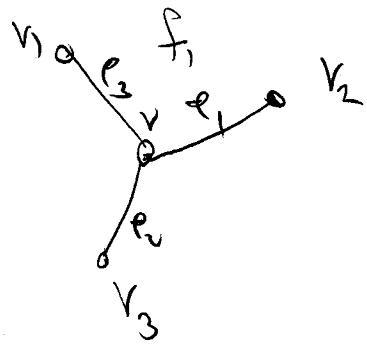
Boundary info:

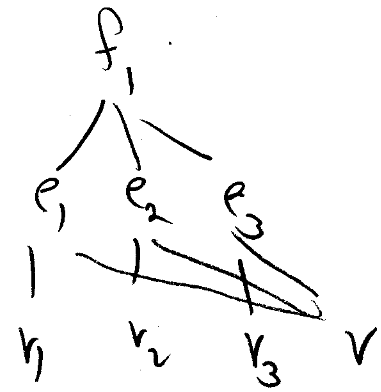
eg





Poset

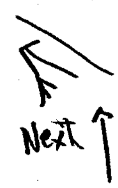




Need
1) order of edges at a vertex
2) edges around a face.

2D solution:
1) doubly-connected edge list
a) write each edge as two arcs

next(arc): arc
prev(arc): arc
reflection(arc) = twin



Group Theoretic Approach

next, prev, reflect are permutation

$A \equiv$ set of arcs

$\varphi \equiv$ next then $\varphi \in Sym(A)$

$\varphi^{-1} \equiv$ prev

$R =$ reflect $R \in Sym(A)$

prop of R fixed-point-free fpf

$R^2 = id$

Def $\sigma_1 \cdots \sigma_k \in$ Group $\langle \sigma_1 \cdots \sigma_k \rangle \equiv$ subgroup generated by $\sigma_1 \cdots \sigma_k$

Orbits$(\langle \sigma_1, \cdots, \sigma_k \rangle)$

$a \equiv b$ iff $\exists \sigma \in \langle \sigma_1 \cdots \sigma_k \rangle$ st $\sigma(a) = b$

Orbits$(\langle R \rangle) \equiv$ edges

Orbits$(\langle \varphi \rangle) \equiv$ faces

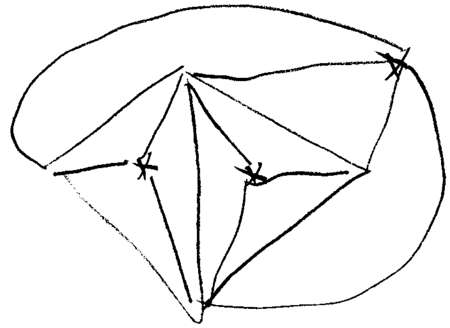
Orbits$\langle \varphi R \rangle \equiv$ vertices

Orbits$(\langle R, \varphi \rangle) \equiv$ connect component

View φ, R acting on triangles.

1) add point in "middle" of each face and form triangles

eg



note 1-1 correspondence arcs & new triangles

φ, R take tri to tri

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another view: Glueing rules:

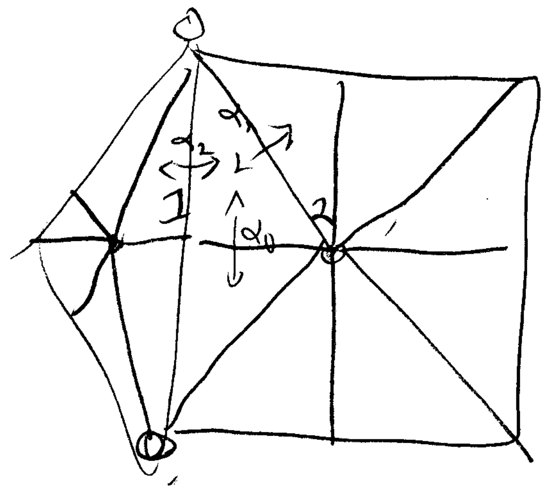
1) Start with # triangles = # arcs

2) Glue triangles, with rules φ, R

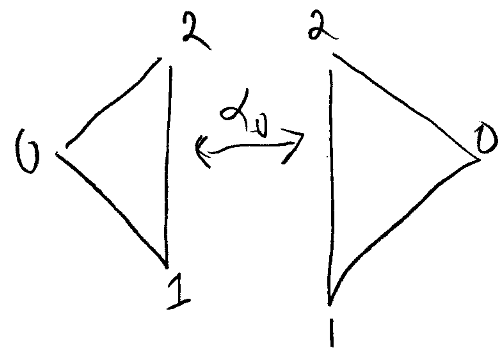
Baricentric Subvision

This construction will work in any dim

1) add a new point in each cell (label by the dim of cell)
2) form simplices out of new points



Switch operators $\alpha_0, \alpha_1, \alpha_2$ (Reflections)



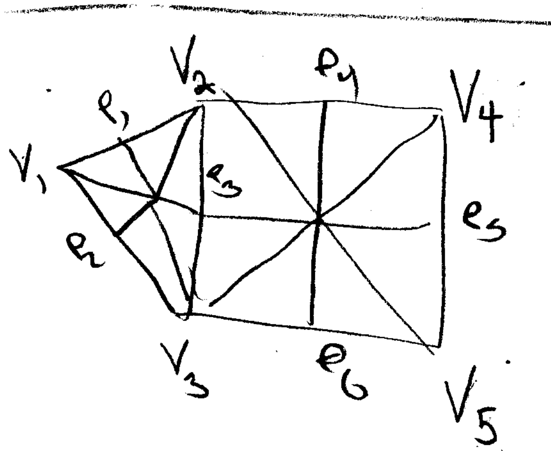
properties α_i is fpf $\quad \alpha_0\alpha_2 = \alpha_2\alpha_0$

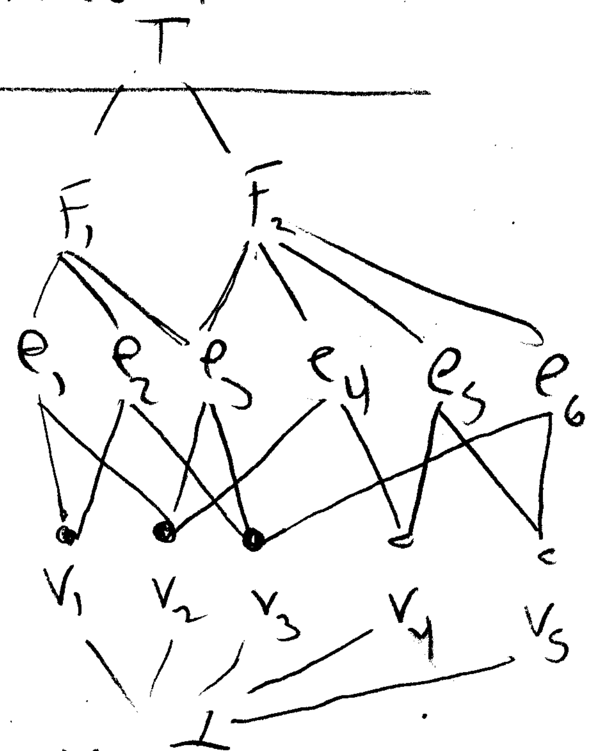
$\alpha_i^2 = id$

Return to Posets

assume:

1) no pinched edges
2) no pinched faces
3) face boundary is connected



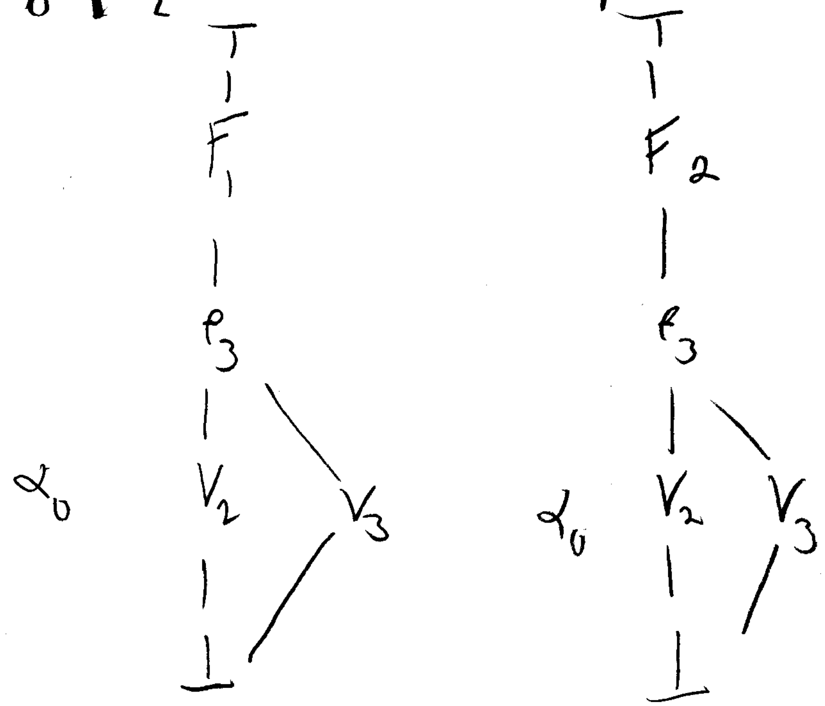


Claim 1-1 correspondence between numbered tris & paths in the Poset

Def cell-tuple (V, E, F) st $V \partial E$ & $E \partial F$.

$$(\perp, V, E, F, \top)$$

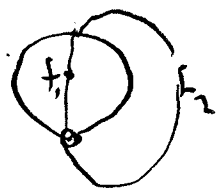
$\alpha_0 \alpha_1 \alpha_2$ act on cell-Tuples



α_0 only depends on triple $(\perp, V, E)$

$\alpha_0(\perp, V, E) = V'$

Claim if we drop 1)



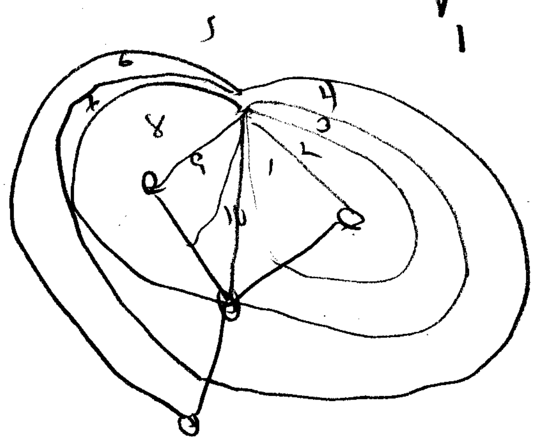
4 numbered Tri

2 tuples

solution add multiple edges

def: path are call ~~the~~ cell-chains

4 cell-chains



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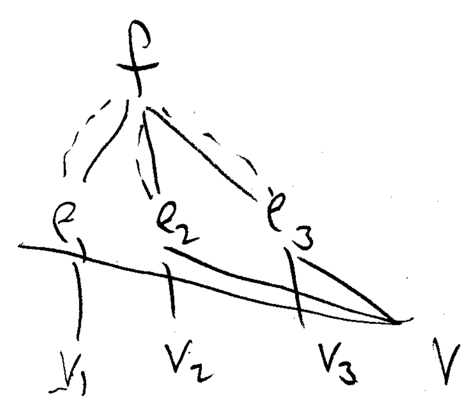

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Formal Structure

Def $M = (\Sigma, \alpha_0, \dots, \alpha_d)$ where (Map)

1) Σ is finite set & $\alpha_0, \dots, \alpha_d \in Sym(\Sigma)$
2) $\alpha_i^2 = id$ for $0 \le i \le d$.
3) α_i fpf for $0 \le i < d$

Note View fp of α_d as boundries.

Def (Commuting-Map) $M = (\Sigma, \alpha_0, \dots, \alpha_d)$

1) M is a map
2) $\alpha_i \alpha_j = \alpha_j \alpha_i$ $\quad i+2 \le j$ (Commutivity)

Def $G_i^j = \langle \alpha_i, \dots, \alpha_j \rangle$ $\quad i \le j$

Def (Cell-Map) $M = (\Sigma, \alpha_0, \dots, \alpha_d)$

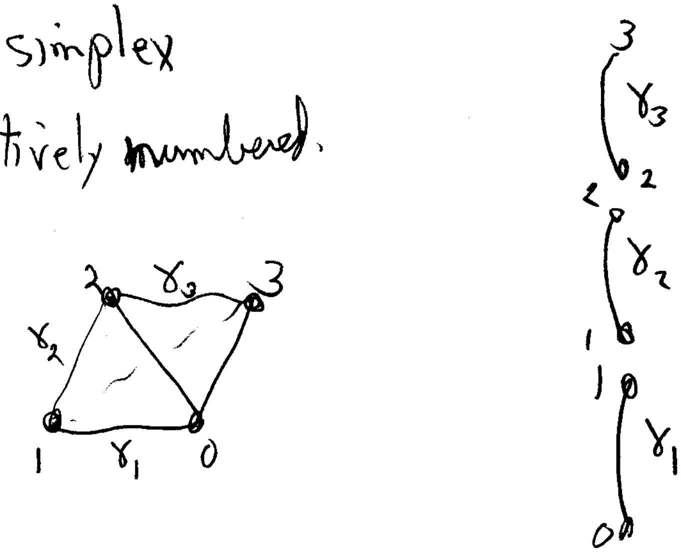
1) M is a Commuting-Map)
2) $\forall i$ $0 < i < d$, $\tau \in G_0^{i-1}$ & $\beta \in G_{i+1}^d$
$\alpha\beta(\tau) = \tau \Rightarrow \alpha(\tau) = \beta(\tau) = \tau$ (Orthogonality)

<u>Def</u> (Cell-Chain) Seq 1-dim numbered simplices

$\gamma_1 \dots \gamma_K$

1) $\gamma_i \cap \gamma_{i+1}$ = 0-simplex
2) Each consecutively numbered.

eg



<u>Thm</u> If M is a cell-map with complex $\mathcal{C}$ then $\exists$ 1-1 correspondence between cell-chains & d-simplices.

<u>Thm</u> If $\mathcal{C}$ is a complex of a commuting-map M then M is a cell-map iff $\exists$ 1-1 between cell-chain & d-simp.