Mesh Generation

Goal: Partition domain into simplices.

∅

Simplex: vertex, segment, triangle, tetrahedron

Partition: Intersection of 2 simplices in a simplex

Conforming to input

Well-shaped simplices

a) no small angles ≥ 0°

b) no large angles ≥ 180°

Small number of simplices (optimal size)
Repeat Ratio

\[ A(a, b, c) = \frac{\text{longest-side}}{\text{alt}} \]

\[ R(a, b, c) = \frac{\text{longest-side}}{\text{shortest-side}} \]

\[ \frac{1}{\text{Smallest-angle}} \]

\[ \frac{1}{180^\circ - \text{largest-angle}} \]

radius-edge ratio = \( \frac{r}{e} \)

\[ r = \text{radius of circum sphere} \]

\[ e = \text{shortest edge} \]
Mesh Generation Methods

1) Quadtree (today)
2) Delaunay Refinement (to do)
3) Advancing Front
4) Ball-Packing
5) Voronoi Refinement

In 2D our input will be PSLG.
Simplex & Simplicial Complex

Def: \( \overline{P_0, \ldots, P_k} \in \mathbb{R}^d \) are affinely independent of dimension \( k \).

If \( P_i - P_0, \ldots, P_k - P_0 \) are independent.

Def: If \( P_0, \ldots, P_k \) are a-ind then \( \operatorname{CC}(P_0, \ldots, P_k) \) is a \( k \)-simplex & \( \forall \{P_0, \ldots, P_k\} \subseteq \operatorname{CC}(S) \) is a sub-simplex

Def: A set \( K \) of simplices in \( \mathbb{R}^d \) is a Simplicial Complex if:

1) \( K \) is closed under sub-simplex

2) \( S, T \in K \) then \( S \cap T \) sub-simplex of \( T \)

Def: \( \dim(K) = \max \{ \dim S \mid S \in K \} \)
Note: $PSLG$ is a 1-dim simplicial complex in $\mathbb{R}^2$.

$K$ & $K'$ are simplicial complexes.

**Def**: $K'$ is a refinement of $K$ if

$\forall s \in K \text{ of dim } k \exists s_1, \ldots, s_k \in K' \text{ of dim } k$

$s \supseteq \bigcup_{i=1}^{k} s_i$

**Input**: Simplicial complex $K$ & Domain $\mathcal{N}$

$s \subseteq K \Rightarrow s \subseteq \mathcal{N}$

**Output**: refinement $K'$ of $K$ s.t.

$\bigcup_{s \in K'} s = \mathcal{N}$
Quad-Tree Meshing

**Input:** set \( X \subseteq \mathbb{R}^2 \) of points \( X \subseteq B \) (box) \( |X| = n \)

**Def** QT is a tree of nested square boxes.

The children of box \( b \) are either:

1) empty (leaf box)
2) 4 children of half the size (split of \( b \))

**Neighbors:** 4 direct neighbors

8 extended neighbors
Def: QT is balanced if every leaf box has no side containing more than one interior node.

[Diagrams showing examples of OK and not OK balanced conditions]
Build-QT(X,B)

Init: QT T = (X,B)

1) While 3 leaf box \((X',b)\) st. b is "crowded"
   split b and assign \(X'\) to new boxes,

2) Balance T by splitting

3) Split all boxes containing a point until it
   has 8 extended neighbors (leaf boxes)
Def: A box \( b \) is crowded if \( \exists x \in b \) and one of the following holds:

1. \( \exists y \neq x \in b \)
2. \( \exists y \in X \) st \( \text{dist}(x, y) \leq 2\sqrt{2} \cdot \text{side length}(b) \)
3. An extended neighbor of \( b \) in split.

Warping:

\( x \in b \) by \( y \) closest corner of \( b \) this warp \( y \).

\( b \) empty & not warped:

\( \Rightarrow \)
Cost to Balance

Thm. T is a QT & T' in its balanced version then $|T'| = O(|T|)$, $|T'| = \# \text{ boxes}$

Prf

Note T is a proper k-ary tree

Note A proper k-ary tree with i internal nodes has size $k^i + 1$ (induct)

Def A box of T or T' is old

Claim A new internal box has an extended neighbor which is old.

Proof by contradiction
Let $b$ be the smallest internal new box with no old ext neigh.

$b_{\text{internal}} \Rightarrow \text{a side of } b \text{ is split twice}$

e.g.

$b'$ is new with no old neigh. \text{ contr.}

\[
\#(T') \leq 8 \cdot n \quad (9n)?
\]

\[
\#(T') \leq 4 \cdot \#(\text{int}(T') +) \leq 32n+1
\]

\[\text{Thm} \quad \text{Balanced QT can be computed in } O(dn)\]

\[\text{time } d = \text{depth}\]