15-456
2/22/13

Mesh Generation

Goal: Partition domain into simplices.

Simplex: $\emptyset$, vertex, segment, triangle, Tetrahedron

Partition: Intersection of 2 simlices is a simplex

Conforming: to input

Well-shaped simplices

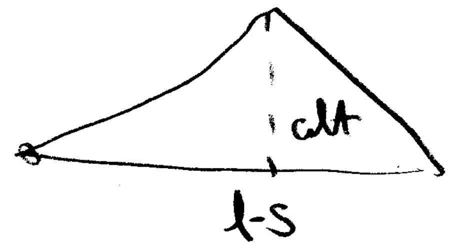
a) no small angles $\approx 0^0$

b) no large angles $\approx 180^0$

Small number of simplices (optimal size)

Aspect Ratio

$$A(a,b,e) = \frac{\text{longest-side}}{\text{alt}}$$



$$R(a,b,c) = \frac{\text{longest-side}}{\text{shortest-side}}$$

$$\frac{1}{\text{smallest-angle}}$$

$$\frac{1}{180^{\circ} - \text{largest-angle}}$$

radius-edge ratio = r/e

r = radius of circum sphere

e = shortest edge

Mesh Generation Methods

1) Quadtree (today)
2) Delaunay Refinement (to do)
3) Advancing Front
4) Ball-Packing
5) Voronoi Refinement

In 2D our input will be PSLG.

Simplex & Simplical Complex

Def: $P_0, \dots, P_k \in \mathbb{R}^d$ are affinely independent of dimension k.

if $P_1 - P_0, \dots, P_k - P_0$ are independent.

Def If $P_0, \dots, P_k$ are a-ind then $CC(P_0, \dots, P_k)$ is a k-simplex & $\forall S \subseteq \{P_0, \dots, P_k\}$ $CC(S)$ is a sub-simplex

Def A set K of simplices in $\mathbb{R}^d$ is a Simplical Complex if:

1) K is closed under sub-simplex
2) $S, \tau \in K$ then $S \cap \tau$ sub-simplex of τ

Def $\dim(K) = \max \dim S \in K$

Note PSLG is a 1-dim simplicial complex in $\mathbb{R}^2$

K & K' are simplical complexes

Def K' is a refinement of K if

$\forall s \in K$ of dim k $\exists s_1, \dots, s_t \in K'$ of dim k

s.t. $S = \bigcup_{i=1}^{t} S_i$

Input: Simplical Complex K & Domain Ω

$s \in K \Rightarrow s \subseteq \Omega$

Output: refinement K' of K s.t $\bigcup_{s \in K'} s = \Omega$

Quad-Tree Meshing

<u>Input</u>: set $X \subseteq \mathbb{R}^2$ of points $X \subseteq B$ (box) $|X| = n$

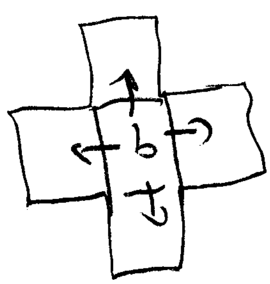
<u>Def</u> QT is a tree of nested square boxes.

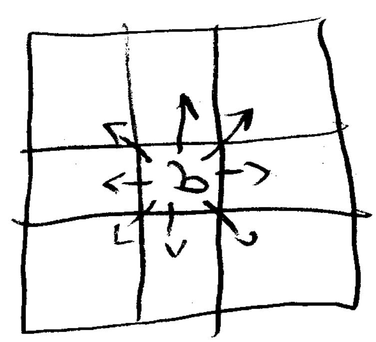
The children of box b are either

1) empty (leaf box)
2) 4 children of half the size (split of b)

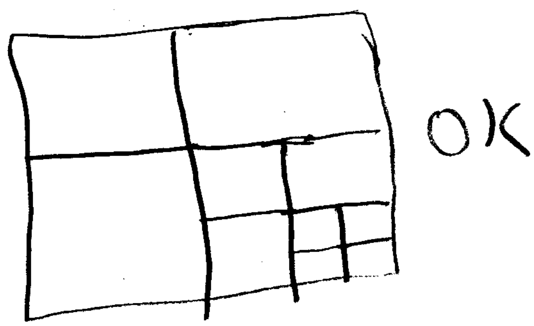
Neighbors: 4 direct neighbors

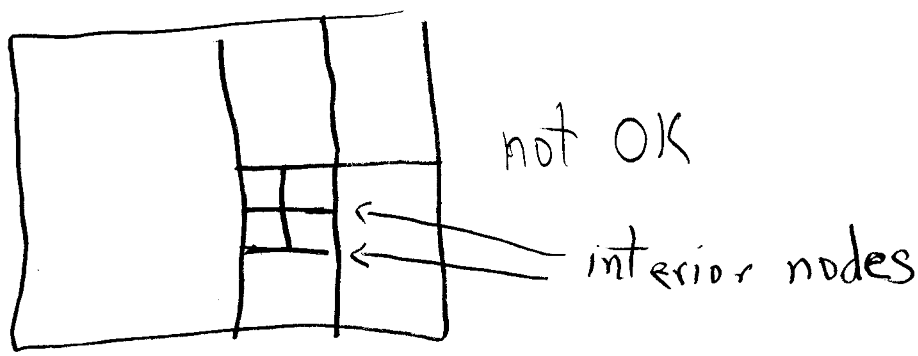
8 extended neighbors





<u>Def</u> QT is <u>balanced</u> if every leaf box has no side containing more than one interior node.





Build-QT(X, B)

Init: QT $T = (X, B)$

1) While $\exists$ leaf box (X', b) s.t. b is "crowded"
   split b and assign X' to new boxes.

2) Balance T by splitting.

3) Split all boxes containing a point until it
   has 8 extented neighbors (leaf boxes).

Def Leaf box b is crowded if $\exists x \in b$ and one of the following holds:

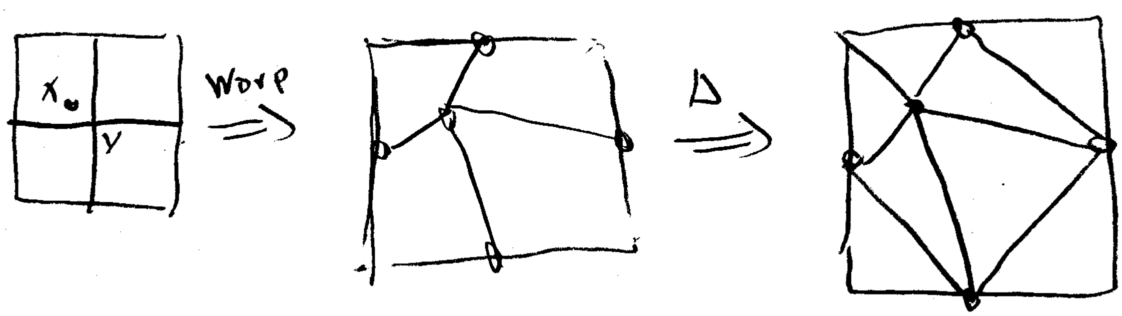
C1) $\exists y \neq x \in b$

C2) $\exists y \in X$ st $\mathrm{dist}(x,y) \le 2\sqrt{2} \cdot \mathrm{sidelength}(b)$

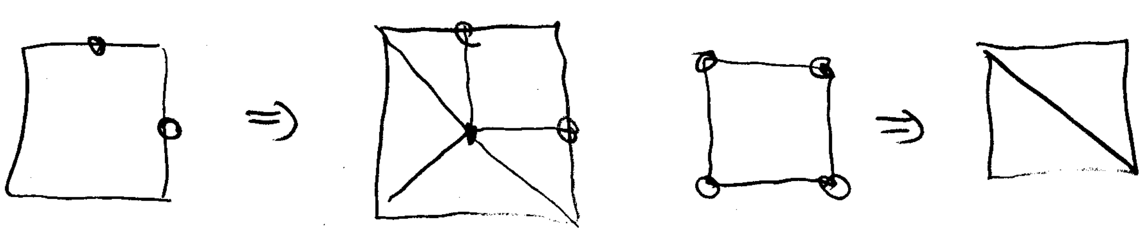
C3) An extended neig of b is split.

Warping

$x \in b$ & y closest corner of b then warp y.



b empty & not warped



Cost to Balance

Thm T is a QT & T' is its balanced version

then $|T'| = O(|T|)$ $|T|$ = # boxes

Pf

note T is a proper 4-ary tree

note A proper k-ary tree with i internal nodes has size $k \cdot i + 1$. (induct)

Def A box of T in T' is old.

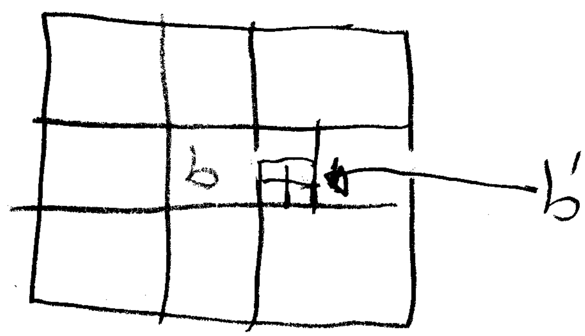
Claim A new internal box has an extended neigh which is old.

Proof by contraction

Let b be the smallest internal new box with no old ext neigh.

b internal $\Rightarrow$ a side of b is split twice.

eg



b' is new with no old neigh. contra!

$\#int(T') \leq 8 \cdot n \quad (9n)?$

$\#(T') \leq 4 \cdot \#(int(T')+) \leq 32n+1$

<u>Thm</u> Balanced QT can be computed in $O(dn)$ time d = depth.