2D only

**Input:** PSLG all angles > 60° (G)

**Output:**
1) 2D simplicial complex
2) A refinement of the PSLG
3) Delaunay
4) no small angle
5) constant times opt size

**Def** \( \text{lfs}(x) = \text{dist to and nearest disjoint feature} \)

**Def** \( f: \mathbb{R}^d \to \mathbb{R} \) is \( \alpha \)-Lipschitz if \( \forall p, q \in \mathbb{R}^d \)

\[ |f(p) - f(q)| \leq \alpha \text{dist}(p, q) \]

**Claim** \( \text{lfs} \) is 1-Lipschitz i.e.

\[ \text{lfs}(p) \leq \text{lfs}(q) + \text{dist}(p, q) \]
Algorithm

Def Circumball of a simplex is min radius ball $B$ with vertices on $DB$.

Def $p$ encroaches simplex $S$ if $p \in \text{Int}(\text{Circumball}(S))$

Def $S'$ encroaches on $S$ if Circumcenter($S'$) $\in \text{Int}(\text{Circumball}(S))$

Def Segment is a subsegment of an edge of $G$.

Delaunay Refinement ($G$) (Overview)

1) Add a bounding box to $G$

2) Compute Delaunay of $V(G) + \text{Box}$

3) While

   1) a segment is encroached add circum-center

   2) a $\Delta$ is skinny add circum-center.
Algorithm Details

Subroutine: Split(segs, S)

1) Add circumcenter of s to V & update DT(V)
2) Remove s from S
3) Add halves of s to S.
Delaunay Refinement \((\text{PSLG } G, \text{ angle } \alpha \text{ or radius-edge } \rho)\)

**Init**

1) Add bounding box to \( G \)
2) \( S = \text{edges}(G) \)
3) \( V = \text{vertices}(G) \)
4) \( T = \text{DT}(V) \)

**While** \( \exists \text{encroached seg or skinny tri do} \)

1) **While** \( \exists \text{seg(s) encroached) split seg(s) do} \)
2) **If** \( t \text{ skinny (radius-edge ) } \rho \text{ then do} \)
   - **If** \( t \text{ encroaches a seg } S \text{ then do} \)
     - **split seg(s)**
   - **else split tri(t)**

**Return** \( \text{DT}(V) \)
Def \[ NN_t(P) \equiv \text{nearest vertex in } V \text{ at last time } P \text{ was considered for insertion before } t. \]

e.g. at step (x) \( P \in \text{Circumcentra}(t) \) was considered but may not have been added.

Def \[ \text{Containing Dimension of } P \equiv \min \text{ dim feature containing } P. \]

e.g. \( P \) is an input point then \( CD(P) = 0 \)

- \( P \) interior to an edge \( CD(P) = 1 \)

- \( P \) vertex \( CD(P) = 2 \)

Lemma \[ \exists \text{ constants } C_\varepsilon \& C_\tau \text{ depending only on } \varepsilon \text{ s.t. } \forall t \]

1) If \( CD(P) = 0 \) then \( lfs(P) \leq NN_t(P) \)

2) If \( CD(P) = 1 \) then \( lfs(P) \leq C_\varepsilon NN_t(P) \)

3) If \( CD(P) = 2 \) then \( lfs(P) \leq C_\tau NN_t(P) \)
Induction on execution time $t$.

Assume $t$ and show $t+1$

**Case** $CD(p) = 0$ ($p$ will only be considered once)

$$\text{hs}_G(p) \leq \text{hs}_V(p) \leq NN_t(p) = NN_{t+1}(p)$$

**Case** $CD(p) = 1$

Let $p \in E_p$ (input edge) & $p$ = circum center of segment $S_p$

$\forall p \in S_p \subset E_p$

$S_p$ must have been encroached by some point $a$

Pick $a \in \text{Ball}(S_p)$ as follows:

1) $a \in \text{Ball}(S_p)$ at time $\text{splitseg}(S_p)$

   set $a$ to closest such point to $p$. 
else let $a \in B(S_p)$ be circumcenter of skinny tri that yielded to $p$.

Subcase $CD(a) = 0$

$$\text{let } a \in B(S_p) \text{ be circumcenter of skinny tri that yielded to } p.$$  

Subcase $CD(a) = 0$

$$\text{let } a \in B(S_p) \text{ be circumcenter of skinny tri that yielded to } p.$$  

Subcase $CD(a) = 0$

$$\text{let } a \in B(S_p) \text{ be circumcenter of skinny tri that yielded to } p.$$  

Subcase $CD(a) = 0$

$$\text{let } a \in B(S_p) \text{ be circumcenter of skinny tri that yielded to } p.$$  

Subcase $CD(a) = 0$

$$\text{let } a \in B(S_p) \text{ be circumcenter of skinny tri that yielded to } p.$$  

Subcase $CD(a) = 0$

$$\text{let } a \in B(S_p) \text{ be circumcenter of skinny tri that yielded to } p.$$  

Subcase $CD(a) = 0$

$$\text{let } a \in B(S_p) \text{ be circumcenter of skinny tri that yielded to } p.$$  

Subcase $CD(a) = 0$

$$\text{let } a \in B(S_p) \text{ be circumcenter of skinny tri that yielded to } p.$$  

Subcase $CD(a) = 0$

$$\text{let } a \in B(S_p) \text{ be circumcenter of skinny tri that yielded to } p.$$  

Subcase $CD(a) = 0$

$$\text{let } a \in B(S_p) \text{ be circumcenter of skinny tri that yielded to } p.$$  

Subcase $CD(a) = 0$

$$\text{let } a \in B(S_p) \text{ be circumcenter of skinny tri that yielded to } p.$$  

Subcase $CD(a) = 0$

$$\text{let } a \in B(S_p) \text{ be circumcenter of skinny tri that yielded to } p.$$  

Subcase $CD(a) = 0$

$$\text{let } a \in B(S_p) \text{ be circumcenter of skinny tri that yielded to } p.$$  

Subcase $CD(a) = 0$

$$\text{let } a \in B(S_p) \text{ be circumcenter of skinny tri that yielded to } p.$$  

Subcase $CD(a) = 0$

$$\text{let } a \in B(S_p) \text{ be circumcenter of skinny tri that yielded to } p.$$  

Subcase $CD(a) = 0$

$$\text{let } a \in B(S_p) \text{ be circumcenter of skinny tri that yielded to } p.$$  

Subcase $CD(a) = 0$

$$\text{let } a \in B(S_p) \text{ be circumcenter of skinny tri that yielded to } p.$$  

Subcase $CD(a) = 0$

$$\text{let } a \in B(S_p) \text{ be circumcenter of skinny tri that yielded to } p.$$  

Subcase $CD(a) = 0$

$$\text{let } a \in B(S_p) \text{ be circumcenter of skinny tri that yielded to } p.$$  

Subcase $CD(a) = 0$

$$\text{let } a \in B(S_p) \text{ be circumcenter of skinny tri that yielded to } p.$$  

Subcase $CD(a) = 0$

$$\text{let } a \in B(S_p) \text{ be circumcenter of skinny tri that yielded to } p.$$  

Subcase $CD(a) = 0$

$$\text{let } a \in B(S_p) \text{ be circumcenter of skinny tri that yielded to } p.$$  

Subcase $CD(a) = 0$

$$\text{let } a \in B(S_p) \text{ be circumcenter of skinny tri that yielded to } p.$$  

Subcase $CD(a) = 0$

$$\text{let } a \in B(S_p) \text{ be circumcenter of skinny tri that yielded to } p.$$  

Subcase $CD(a) = 0$

$$\text{let } a \in B(S_p) \text{ be circumcenter of skinny tri that yielded to } p.$$  

Subcase $CD(a) = 0$

$$\text{let } a \in B(S_p) \text{ be circumcenter of skinny tri that yielded to } p.$$  

Subcase $CD(a) = 0$

$$\text{let } a \in B(S_p) \text{ be circumcenter of skinny tri that yielded to } p.$$  

Subcase $CD(a) = 0$

$$\text{let } a \in B(S_p) \text{ be circumcenter of skinny tri that yielded to } p.$$  

Subcase $CD(a) = 0$

$$\text{let } a \in B(S_p) \text{ be circumcenter of skinny tri that yielded to } p.$$  

Subcase $CD(a) = 0$

$$\text{let } a \in B(S_p) \text{ be circumcenter of skinny tri that yielded to } p.$$  

Subcase $CD(a) = 0$

$$\text{let } a \in B(S_p) \text{ be circumcenter of skinny tri that yielded to } p.$$  

Subcase $CD(a) = 0$

$$\text{let } a \in B(S_p) \text{ be circumcenter of skinny tri that yielded to } p.$$  

Subcase $CD(a) = 0$

$$\text{let } a \in B(S_p) \text{ be circumcenter of skinny tri that yielded to } p.$$  

Subcase $CD(a) = 0$

$$\text{let } a \in B(S_p) \text{ be circumcenter of skinny tri that yielded to } p.$$  

Subcase $CD(a) = 0$

$$\text{let } a \in B(S_p) \text{ be circumcenter of skinny tri that yielded to } p.$$  

Subcase $CD(a) = 0$

$$\text{let } a \in B(S_p) \text{ be circumcenter of skinny tri that yielded to } p.$$  

Subcase $CD(a) = 0$

$$\text{let } a \in B(S_p) \text{ be circumcenter of skinny tri that yielded to } p.$$  

Subcase $CD(a) = 0$

$$\text{let } a \in B(S_p) \text{ be circumcenter of skinny tri that yielded to } p.$$  

Subcase $CD(a) = 0$

$$\text{let } a \in B(S_p) \text{ be circumcenter of skinny tri that yielded to } p.$$  

Subcase $CD(a) = 0$

$$\text{let } a \in B(S_p) \text{ be circumcenter of skinny tri that yielded to } p.$$  

Subcase $CD(a) = 0$

$$\text{let } a \in B(S_p) \text{ be circumcenter of skinny tri that yielded to } p.$$  

Subcase $CD(a) = 0$

$$\text{let } a \in B(S_p) \text{ be circumcenter of skinny tri that yielded to } p.$$  

Subcase $CD(a) = 0$

$$\text{let } a \in B(S_p) \text{ be circumcenter of skinny tri that yielded to } p.$$  

Subcase $CD(a) = 0$

$$\text{let } a \in B(S_p) \text{ be circumcenter of skinny tri that yielded to } p.$$  

Subcase $CD(a) = 0$

$$\text{let } a \in B(S_p) \text{ be circumcenter of skinny tri that yielded to } p.$$  

Subcase $CD(a) = 0$

$$\text{let } a \in B(S_p) \text{ be circumcenter of skinny tri that yielded to } p.$$  

Subcase $CD(a) = 0$

$$\text{let } a \in B(S_p) \text{ be circumcenter of skinny tri that yielded to } p.$$  

Subcase $CD(a) = 0$

$$\text{let } a \in B(S_p) \text{ be circumcenter of skinny tri that yielded to } p.$$  

Subcase $CD(a) = 0$

$$\text{let } a \in B(S_p) \text{ be circumcenter of skinny tri that yielded to } p.$$  

Subcase $CD(a) = 0$

$$\text{let } a \in B(S_p) \text{ be circumcenter of skinny tri that yielded to } p.$$  

Subcase $CD(a) = 0$

$$\text{let } a \in B(S_p) \text{ be circumcenter of skinny tri that yielded to } p.$$  

Subcase $CD(a) = 0$

$$\text{let } a \in B(S_p) \text{ be circumcenter of skinny tri that yielded to } p.$$  

Subcase $CD(a) = 0$

$$\text{let } a \in B(S_p) \text{ be circumcenter of skinny tri that yielded to } p.$$  

Subcase $CD(a) = 0$

$$\text{let } a \in B(S_p) \text{ be circumcenter of skinny tri that yielded to } p.$$  

Subcase $CD(a) = 0$

$$\text{let } a \in B(S_p) \text{ be circumcenter of skinny tri that yielded to } p.$$  

Subcase $CD(a) = 0$

$$\text{let } a \in B(S_p) \text{ be circumcenter of skinny tri that yielded to } p.$$  

Subcase $CD(a) = 0$

$$\text{let } a \in B(S_p) \text{ be circumcenter of skinny tri that yielded to } p.$$  

Subcase $CD(a) = 0$

$$\text{let } a \in B(S_p) \text{ be circumcenter of skinny tri that yielded to } p.$$  

Subcase $CD(a) = 0$

$$\text{let } a \in B(S_p) \text{ be circumcenter of skinny tri that yielded to } p.$$  

Subcase $CD(a) = 0$
Subcase \( \text{CD}(a) = 2 \)

By induction \( \ellfs(a) \leq C_t \ NN_t(a) \leq C_t r' \)

1) \( \NN_t(p) = r \) (will yield to \( S_p \))
2) \( a \in B(p, r) \) & \( x, y \in B(a, r') \) \( \Rightarrow r' \leq \sqrt{2} r \)

\( \ellfs(p) \leq \ellfs(a) + \text{dist}(p, a) \leq C_t r' + r \)
\( \leq C_t \sqrt{2} r + r \)
\( \leq (\sqrt{2} C_t + 1) \ NN_{t+1}(p) \)

We need \( 1 + \sqrt{2} C_t \leq C_e \)
Case \( CD(p) = 2 \)

WLOG \( a \) added before \( b \).

Subcase \( CD(b) = 0 \)

\[ \text{ltS}(p) = r = NN_{th}(p) \]

\[ 1 \leq C_t \]

Note \( CD(b) = 0 \)

\[ \Rightarrow CD(a) = 0 \]

\[ \Rightarrow \text{ltS}(p) = r \]

Subcase \( CD(b) = 1 \)

(induct) \( \text{ltS}(b) \leq C_e NN_{t}(b) \leq C_e \text{dist}(b, a) \)

radio-edge \( e \leq \frac{r}{\rho} \) i.e. \( e \leq \frac{r}{\rho} \)

\[ C_e \frac{r}{\rho} \]

\[ e \leq \frac{r}{\rho} \]

\[ \text{ltS}(p) \leq \text{ltS}(b) + \text{dist}(p, b) = \text{ltS}(b) + r \]

\[ \leq C_e \text{dist}(b, a) + r \]

\[ < C_e \frac{r}{\rho} + r = (C_e \frac{r}{\rho} + 1) r \]

\[ = (C_e \frac{r}{\rho} + 1) NN_{th}(p) \]

need \( (C_e \frac{r}{\rho} + 1) \leq C_t \)
Subcase, CD(b) = 2

Same as last case but \( C_e \) is \( C_t \)

i.e. \( (C_t \bar{\rho} + 1) \leq C_t \)

\[
\frac{1}{1-\rho} \leq C_t \quad \text{or} \quad \frac{\rho}{\rho-1} \leq C_t
\]

Our list of needed conditions for \( \rho \)

1. \( \frac{1}{2} \leq C_e \)
2. \( \frac{\rho}{\rho-1} \leq C_t \)
3. \( 1 + \sqrt{2} C_t \leq C_e \)
4. \( 1 + \bar{\rho} C_e \leq C_t \)
\( C_t = \frac{C_e}{\sqrt{2}} - \frac{1}{\sqrt{2}} \sqrt{\frac{1}{2}} \) 

\( C_t = \bar{p} \left( C_e + 1 \right) \) \text{ slope } \bar{p} \Rightarrow \bar{p} \leq \frac{1}{\sqrt{2}} 

on \bar{p} = \sqrt{2}
\[ p = \frac{\sqrt{2}}{e} \geq \sqrt{2} \]

\[ \frac{\frac{\sqrt{2}}{r}}{r} = \sin \alpha \]

\[ \left( \frac{1}{2\sqrt{2}} \right) = \sin \alpha \]

\[ \Rightarrow \alpha \approx \sin^{-1} \left( \frac{1}{2\sqrt{2}} \right) \approx 20^0 \]
Thm: \( \text{Output then } \text{lfs}(p) \leq (c_e+1) \text{NN}_{\text{output}}(p) \)

pf:

Let \( q \) be NN of \( p \) in output i.e. \( \text{NN}_{\text{output}}(p) = \text{dist}(p, q) \)

Case 1: \( p \) added after \( q \) then \( \text{NN}_{+}(p) = 1 \times \text{NN}_{\text{output}}(p) \)

\[ \text{lfs}(p) \leq c_e \text{NN}_{+}(p) = c_e \text{NN}_{\text{output}}(p) \]

Case 2: \( q \) added after \( p \).

\[ \text{NN}_{+}(q) \leq \text{dist}(p, q) \]

\[ \text{lfs}(p) \leq \text{lfs}(q) + \text{dist}(p, q) \leq c_e \text{NN}_{+}(q) + \text{dist}(p, q) \]

\[ \leq (c_e+1) \text{dist}(p, q) \]

\[ = (c_e+1) \text{NN}_{\text{output}}(p) \]
Thm: DR generates a mesh with at most
\[ C \int_{\Box x} \frac{1}{\text{afs}(x)} \, dA \] vertices.

Proof: Let \( r_p = \frac{\text{afs}(p)}{2(C_\text{c+1})} \).

Note: Balls \( B(p, r_p) \) are disjoint.

Note: \( \max_{x \in B_p} \text{afs}(x) \leq \text{afs}(p) + r_p \)

\[ \int_{B_p} \frac{1}{\text{afs}(x)} \, dA \geq \text{Area}(B_p) \frac{1}{(\text{afs}(p) + r_p)^2} = \frac{\pi r_p^2}{(2(C_\text{c+1})r_p + r_p)^2} \]

\[ = \frac{\pi}{(2(C_\text{c+3})^2) \equiv C'} \]
\[
\int_{\text{Box}} \frac{1}{\mathcal{A}^2(x)} \, dA = \sum_{p \in V(D)} \frac{1}{4} \int_{B_p} \frac{1}{\mathcal{A}^2(x)} \, dA
\]
\[
\geq \sum_{p \in V} \frac{1}{4} c' = \frac{1}{4} c' |V|
\]