Delaunay Refinement

Ruppert

2D only

Input: PSLG all angles > 60° (G)

Output:
1) 2D simplicial complex
2) A refinement of the PSLG
3) Delaunay
4) no small angle
5) constant times opt size

Def $lfs(x) \equiv$ dist to 2nd nearest disjoint feature.

Def $f: R^d \to R$ is α-Lipschitz if $\forall p, q \in R^d$

$$|f(p) - f(q)| \le \alpha \, dist(p, q)$$

Claim lfs is 1-Lipschitz i.e.

$$lfs(p) \le lfs(q) + dist(p, q)$$

Algorithm

Def Circumball of a simplex is min radius ball B with vertices on ∂B.

Def p encroaches simplex S if $p \in \mathrm{Inter}(\mathrm{Circumball}(S))$
S' encroaches on S if $\mathrm{circumcenter}(S') \in \mathrm{Inter}(\mathrm{Circumball}\, S)$

Def Segment is a subsegment of an edge of G.

Delaunay Refinement (G) (Overview)

1) Add a bounding box to G
2) Compute Delaunay of $V(G)$+Box
3) While
   1) a segment is encroached add circum-center
   2) a Δ is skinny add circum-center.

Algorithm Details

Subroutine: Split(seg s, S)

1) Add circumcenter of s to V & update DT(V)

2) a) remove s from S

b) add halves of s to S.

Delaunay Refinement (PSLG G, angle α or radius-edge ρ)

Init
1) Add bding box to G
2) S = edges(G)
3) V = vertices(G)
4) T = DT(V)

While $\exists$ encroached seg or skinny tri do

1) While $\exists s \in S$ (s encroached) splitseg(s)
2) If t skinny (radius-edge > ρ) then do (*)
   If t encroaches a seg s then do
   splitseg(s)
   else splittri(t)

Return DT(V)

Def $NN_t(p) \equiv$ nearest vertex in V at last time
p was considered for insertion before t.

e.g. at step (*) $p \equiv$ circumcenter(t) was considered but may not have been added.

Def Containing Dimension of $p \equiv$
min dim feature containing p.

eg p is an input point then $CD(p)=0$
p interior to an edge $CD(p)=1$
o.w. $CD(p)=2$

Lemma $\exists$ constants C_e & C_t depending only on α s.t. $\forall t$
1) If $CD(p)=0$ then $lfs(p) \leq NN_t(p)$
2) If $CD(p)=1$ then $lfs(p) \leq C_e NN_t(p)$
3) If $CD(p)=2$ then $lfs(p) \leq C_t NN_t(p)$

pf Induction on execution time, t.

Assume t and show $t+1$

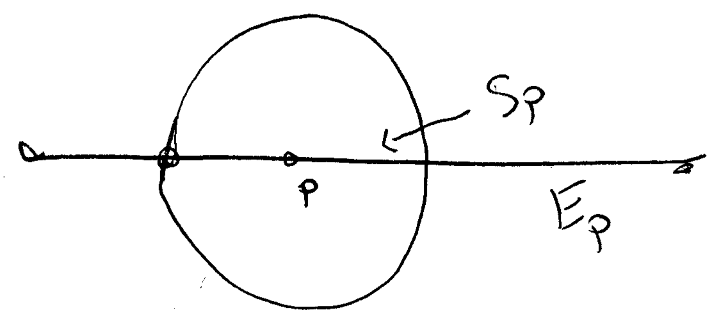
Case $CD(p)=0$ (p will only be considered once)

$$lfs_G(p) \leq lfs_V(p) \leq NN_t(p) = NN_{t+1}(p)$$

Case $CD(p)=1$

Let $p \in E_p$ (input edge) & p = circumcenter of segment S_p

ie $p \in S_p \subseteq E_p$



S_p must have been encroached by some point a

Pick $a \in Ball(S_p)$ as follows.

1) $\exists a \in Ball(S_p)$ at time splitseg(S_p)

set a to closest such point to p.

else let $a \in B(S_p)$ be circumcenter of skinny tri that yielded to p.

Subcase $CD(a)=0$

$$\underbrace{lfs(p) \leq dist(p,a)}_{CD(a)=0} \leq NN_{t+1}(p)$$

a was closest point.

we need $\boxed{1 \leq C_e}$

Subcase $CD(a)=1$

Let $a \in E_a$ (input edge)

(Let's assume input angles $\geq 90°$)

Thus $E_a \cap E_p = \emptyset$

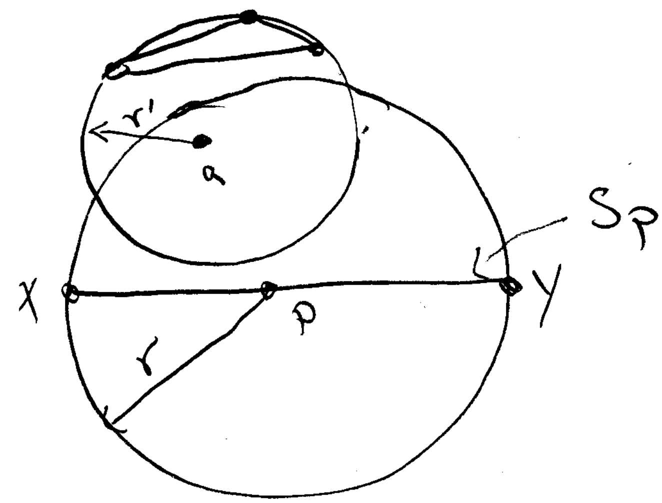
$$lfs(p) \leq dist(p, E_a) \leq dist(p,a) = NN_{t+1}(p)$$

$$\boxed{1 \leq C_e}$$

($60°$ case?)

Subcase $CD(a) = 2$

By induction $lfs(a) \le C_t \, NN_t(a) \le C_t r'$

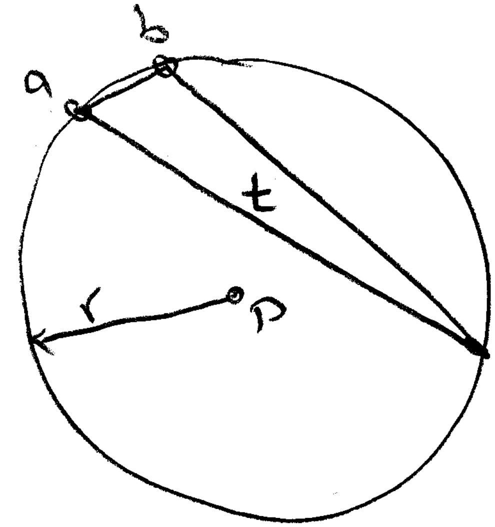


---

1) $NN_t(p) = r$ (a will yeild to S_p)
2) $a \in B(p, r)$ & $x, y \notin B(a, r') \Rightarrow r' \le \sqrt{2}\, r$

$$
\begin{aligned}
lfs(p) \le lfs(a) + dist(p, a) &\le C_t r' + r \\
&\le C_t \sqrt{2}\, r + r \\
&\le (\sqrt{2}\, C_t + 1)\, NN_{t+1}(p)
\end{aligned}
$$

We need $\boxed{1 + \sqrt{2}\, C_t \le C_e}$



Case $CD(p)=2$

WLOG a added before b.

Subcase $CD(b)=0$

$lfs(p) = r = NN_{t+1}(p)$ $\boxed{1 \le C_t}$

Note $CD(b)=0$
$\Rightarrow CD(a)=0$
$\Rightarrow lfs(p)=r$

Subcase $CD(b)=1$

(induct) $lfs(b) \le C_e NN_t(b) \le C_e \, dist(b,a)$

radius-edge $\rho \le r/e$ ie $e \le r/\rho$ $\quad \bar{\rho} = 1/\rho$
$e \le \bar{\rho} r$

$$\begin{aligned} lfs(p) &\le lfs(b) + dist(p,b) = lfs(b) + r \\ &\le C_e \, dist(b,a) + r \\ &< C_e \bar{\rho} r + r = (C_e \bar{\rho} + 1) r \\ &= (C_e \bar{\rho} + 1) NN_{t+1}(p) \end{aligned}$$

need $\boxed{(C_e \bar{\rho} + 1) \le C_t}$

Subcase $CD(b)=2$

Same as last case but C_e is C_t

ie $(C_t\bar{\rho}+1) \le C_t$

$$\frac{1}{1-\bar{\rho}} \le C_t \quad \text{or} \quad \frac{\rho}{\rho-1} \le C_t$$

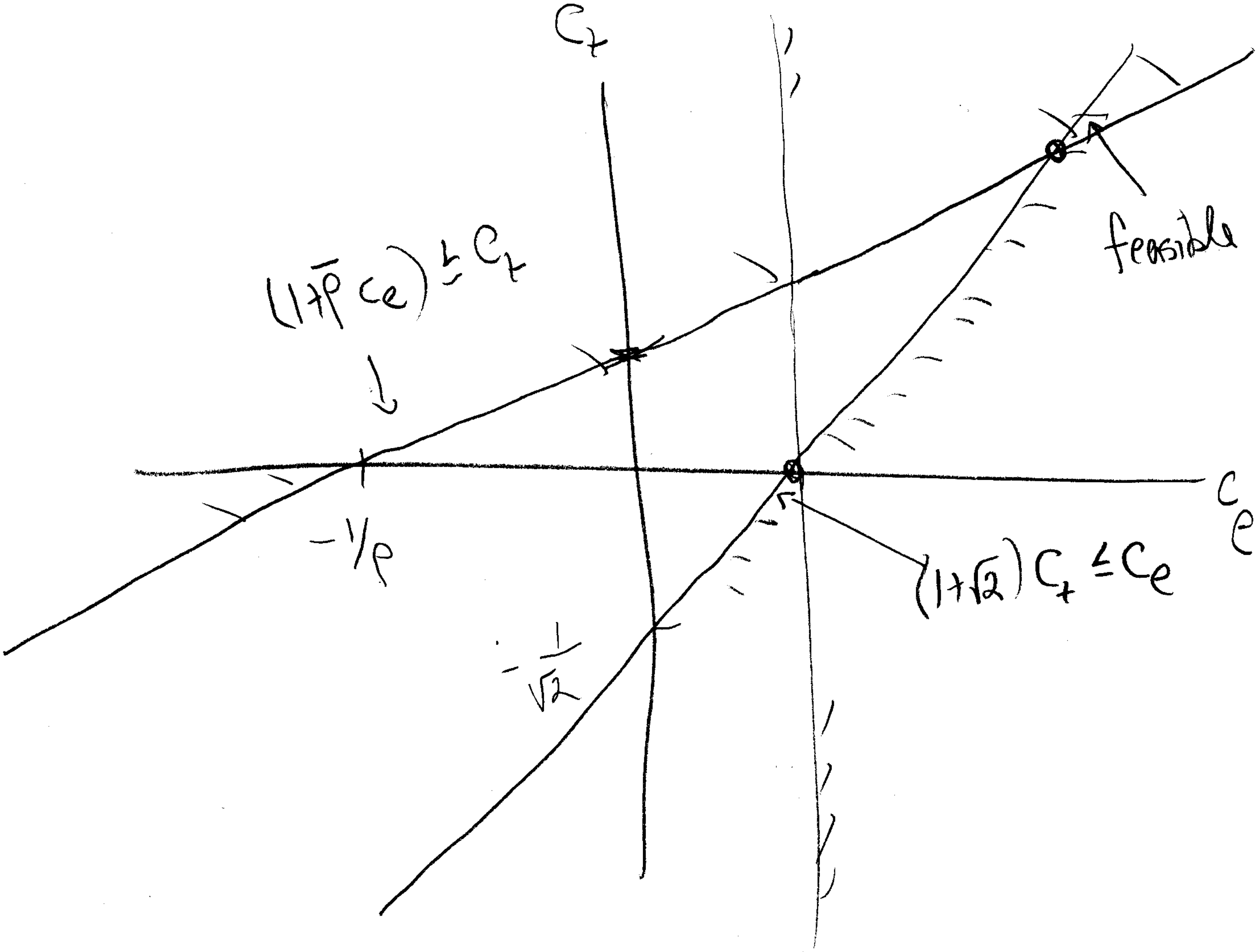
Our list of needed conditions for ρ

$1 \le C_e$

$\frac{\rho}{\rho-1} \le C_t$

$1+\sqrt{2}\,C_t \le C_e$

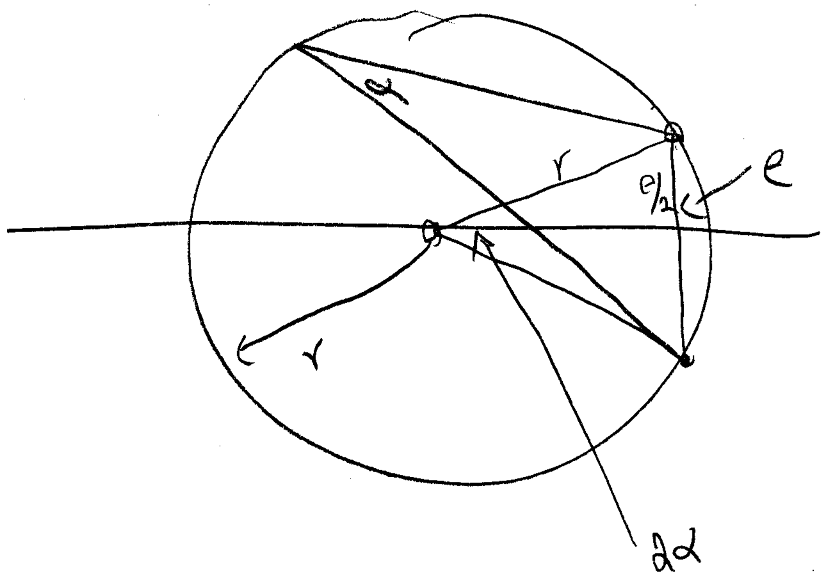
$1+\bar{\rho}\,C_e \le C_t$



$$\left.\begin{array}{ll} C_t = \frac{C_e}{\sqrt{2}} - \frac{1}{\sqrt{2}} & \text{slope } \frac{1}{\sqrt{2}} \\ C_t = \bar{\rho}\, C_e + 1 & \text{slope } \bar{\rho} \end{array}\right\} \Rightarrow \bar{\rho} \le \frac{1}{\sqrt{2}}$$

or $\rho \ge \sqrt{2}$

$\rho = r/e \geq \sqrt{2}$



$\frac{e/2}{r} = \sin\alpha$

$\left(\frac{1}{2\sqrt{2}}\right) = \sin\alpha \quad \Rightarrow \quad \alpha \approx \sin^{-1}\left(\frac{1}{2\sqrt{2}}\right) \approx 20^0$

<u>Thm</u> $p \in$ Output then $lfs(p) \le (c_e+1)\, NN_{output}(p)$

<u>pf</u>

Let q be NN of p in output ie $NN_{output}(p) = dist(p,q)$

<u>Case 1</u> p added after q then $NN_t(p) = NN_{output}(p)$

$$lfs(p) \le c_e\, NN_t(p) = c_e\, NN_{output}(p)$$

<u>Case 2</u> q added after p.

$$NN_t(q) \le dist(p,q)$$

$$lfs(p) \le lfs(q) + dist(p,q) \le c_e\, NN_t(q) + dist(p,q)$$

$$\le (c_e+1)\, dist(p,q)$$

$$= (c_e+1)\, NN_{output}(p)$$

Thm DR generates a mesh with at most

$$C\int_{B_{1x}} \frac{1}{\mathrm{lfs}^2(x)}\, dA \text{ vertices.}$$

pf Let $r_p = \dfrac{\mathrm{lfs}(p)}{2(C_e+1)}$

Note Balls $B(p, r_p)$ are disjoint.
$$\underset{B_p}{\overset{\shortparallel}{B(p, r_p)}}$$

Note $\displaystyle\max_{x\in B_p} \mathrm{lfs}(x) \le \mathrm{lfs}(p) + r_p$

$$\int_{B_p} \frac{1}{\mathrm{lfs}^2(x)}\, dA \geq \mathrm{Area}(B_p)\, \frac{1}{(\mathrm{lfs}(p)+r_p)^2} = \frac{\pi r_p^2}{\left(2(c_e+1)r_p + r_p\right)^2}$$

$$= \frac{\pi}{(2c_e+3)^2} \equiv C'$$

$$\int_{Box} \frac{1}{lfs^2(x)}\,dA \geq \sum_{p \in V(D)} \frac{1}{4} \int_{B_p} \frac{1}{lfs^2(x)}\,dA$$

$$\geq \sum_{p \in V} \frac{1}{4} C' = \frac{1}{4} C' |V|$$