13-456

4/22/13

de Casteljau Alg, Bezier Curves
Bernstein Polys

eg

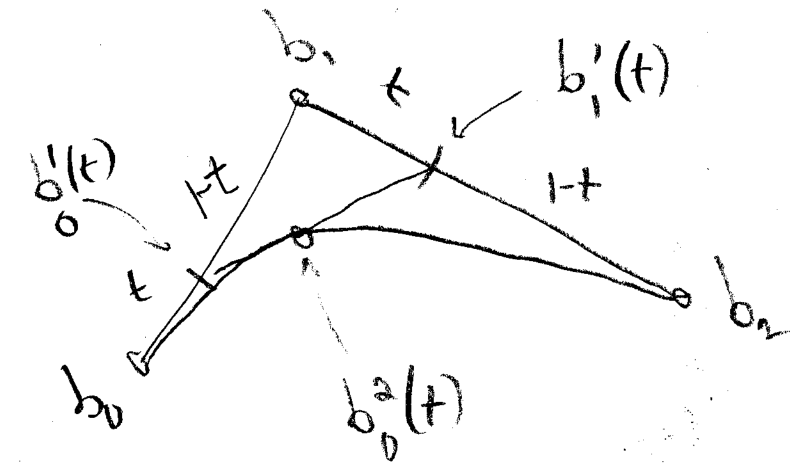
Let $b_0, b_1, b_2 \in \mathbb{R}^2$ & $0 \le t \le 1$

$$b_0^1(t) = (1-t)\, b_0 + t\, b_0$$

$$b_1^1(t) = (1-t)\, b_1 + t\, b_2$$

$$b_0^2(t) = (1-t)\, b_0^1(t) + t\, b_1^1(t)$$

picture



As a Polynomial

$$b_0^2(t) = (1-t)\big((1-t)b_0 + t\,b_1\big) + t\big((1-t)b_1 + t\,b_2\big)$$

$$= \underbrace{(1-t)^2}_{\alpha} b_0 + \underbrace{2t(1-t)}_{\beta} b_1 + \underbrace{t^2}_{\gamma} b_2$$

$$\alpha + \beta + \gamma = \big((1-t) + t\big)^2 = 1$$

de Castaljan Alg

$b_0 \cdots b_n \in \mathbb{R}^d \quad t \in \mathbb{R}$

set $\quad b_i^r = (1-t) b_i^{r-1}(t) + t\, b_{i+1}^{r-1}(t) \qquad r = 1, \cdots n \quad i = 0, \cdots n-r$

b_i^r is poly curve of degree r

<u>Claim</u> de Casteljau commutes with affine maps

ie $X \to AX+b$

$$\begin{tikzcd} \cdot \arrow[r, "Af"] \arrow[d, "Cast"'] & \cdot \arrow[d, "Cast"] \\ \cdot \arrow[r, "Af"] & \cdot \end{tikzcd}$$

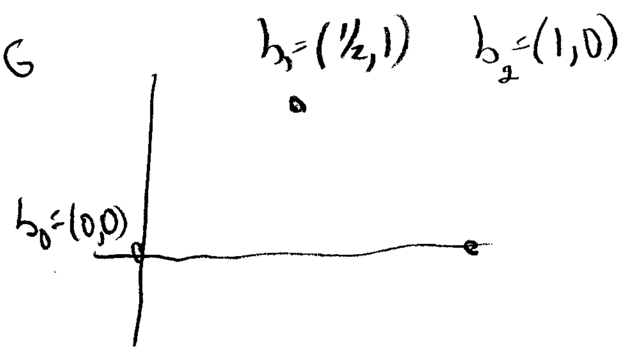
$$b_0^1(t) = (1-t)b_0 + tb_1$$

$$= (1-t)(Ab_0+b) + t(Ab_1+b)$$

$$= A((1-t)(b_0)) + (1-t)b + Atb_1 + tb)$$

$$A((1-t)b_0 + tb_1) + b$$

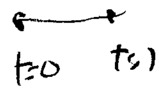
WLOG



Representing Curves
Implicit & Parametric

Parametric

Line $\to \mathbb{R}^2$

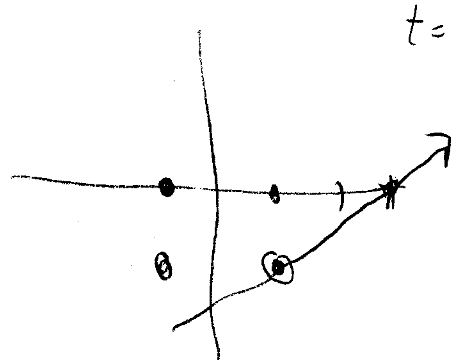


$x = 2t+1$

$y = t-1$

$t=0 \quad (1,-1)$

$t=1 \quad (3,0)$



Implicit

$$x^2+y^2-1=0$$

Parametric

$x = \cos\theta$

$y = \sin\theta$

Implicit $ax^2+bxy+cy^2+dx+ey+f=0$

Parabolas $y^2 = 4ax$

Ellipses $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ Circles $a=b$

Hyperbolas $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$

Polynomial Curves of degree 1 & 2

Degree 1

$$x(t) = a_1 t + a_0$$

$$y(t) = b_1 t + b_0$$

$a_1 \cdot b_1 = 0$ then curve is horizonal or vertical line

$$b_1 x = a_1 b_1 t + a_0 b_1$$

$$a_1 y = a_1 b_1 t + a_1 b_0$$

$$\boxed{b_1 x - a_0 b_1 = a_1 y - a_1 b_0}$$ linear implicit

Degree 2

$$x(t) = F_1(t) = a_2 t^2 + a_1 t + a_0$$

$$y(t) = F_2(t) = b_2 t^2 + b_1 t + b_0$$

$a_2 \neq 0$ or $b_2 \neq 0$ $\quad \rho = \sqrt{a_2^2 + b_2^2}$

$$R = \begin{pmatrix} b_2/\rho & -a_2/\rho \\ \frac{a_2}{\rho} & b_2/\rho \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = R \begin{pmatrix} x \\ y \end{pmatrix}$$ change of variables

$$X_1(t) = \left(\frac{b_2 a_2}{\rho} t^2 + \cdots \quad - \frac{a_2 b_2}{\rho} t^2 + \cdots \right)$$

$$X_1(t) = a_1' t + a_0'$$
$$Y_1(t) = b_2' t^2 + b_1' t + b_0' \quad b_2' \neq 0$$

$a_1' = 0$ degenerate

goal eliminate b_1'

$$u = t + \frac{b_1'}{2b_2'}$$

Our form is

$$x(u) = a_1 u + a_0$$
$$y(u) = b_2 u^2 + b_0$$

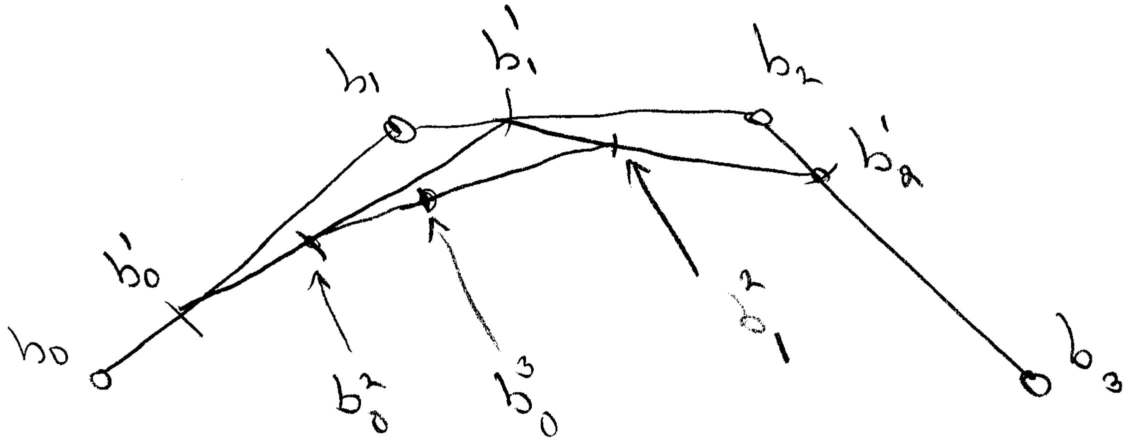
a translation

$$x(u) = a u$$
$$y(u) = b u^2 \quad b > 0$$

$$\frac{b}{a^2}x^2 = bu^2 = y$$

Thm Parametric Quad are parabolas

no Ellipses or Hyperbolas



fixed t

$$
\begin{array}{llll}
b_0 & & & \\
b_1 & b_0^1 & & \\
b_2 & b_1^1 & b_0^2 & \\
b_3 & b_2^1 & b_1^2 & b_0^3
\end{array}
$$

Computing Bezier by Subdivision

first $b_0, \ldots, b_n$

$$
\begin{array}{lllll}
b_0 & & & & \\
b_1 & b_0^1 & & & \\
\vdots & \vdots & b_0^2 & \ddots & \\
\vdots & & \vdots & & \\
b_n & b_{n-1}^1 & b_{n-2}^2 & \cdots & b_0^n
\end{array}
$$

2 control polygons (polygonal paths)

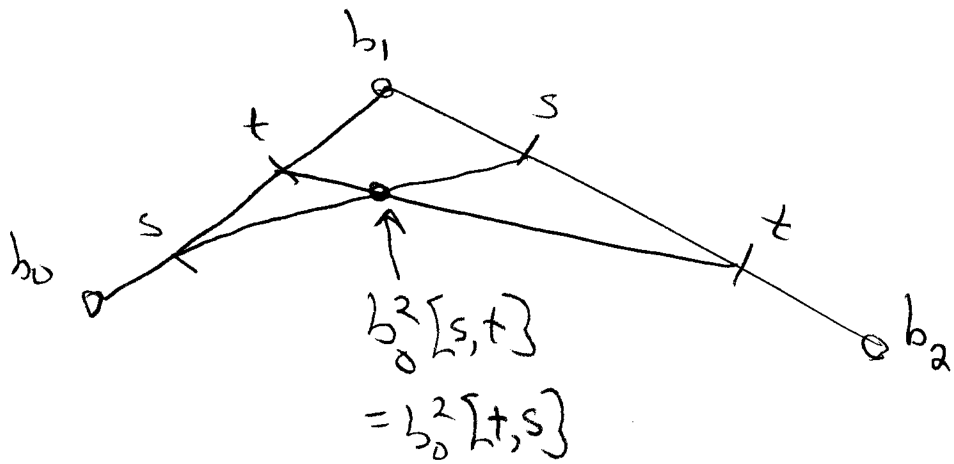
$$(b_0\, b_0^1 \cdots b_0^n) \;\&\; (b_0^n\, b_1^{n-1} \cdots b_n^0)$$

<u>Claim</u> $\underset{0 \le s \le 1}{(b_0, \ldots, b_n)} \equiv$

$$\underset{0 \le s \le t}{(b_0\, b_0^1 \cdots b_0^n)} * \underset{t \le s \le 1}{(b_0^n\, b_1^{n-1} \cdots b_n^0)}$$

Same curve!

Blossoms



b_0

$b_1 \quad b_0^1[s]$

$b_2 \quad b_1^1[s] \quad b_0^2[s,t]$

b_0

$b_1 \quad b_0^1[t]$

$b_2 \quad b_1^1[t] \quad b_0^2[t,s]$

Menelao's Thm $\quad b_0^2[t,s] = b_0^2[s,t]$

b_0

$b_1 \quad b_0^1[t_1]$

$b_2 \quad b_1^1[t_1] \quad b_0^2[t_1, t_2]$

$b_3 \quad b_2^1[t_1] \quad b_1^2[t_1, t_2] \quad b_0^3[t_1, t_2\, t_3]$

<u>Claim</u> $b[0,0,0] = b_0$

$b[1,1,1] = b_3$

$b[1,0,0] = b[0,1,0] = b[0\,0\,1] = b_1$

$b_2[1,1,0] = \qquad = b_2$

$b_0 = b[0,0,0]$

$b_1 = b[0\,0\,1] \quad b[0\,0\,t]$

$b_2 = b[0\,1\,1] \quad b[0\,t\,1] \quad b[0\,t\,t]$

$b_3 = b[1,1,1] \quad b[t,1\,1] \quad b[t\,t\,1] \quad b[t\,t\,t]$

Bernstein Polynomials

Bernstein: $B_i^n(t) = \binom{n}{i} t^i (1-t)^{n-i}$

$$\binom{n}{i} = \begin{cases} \dfrac{n!}{i!\,(n-i)!} & 0 \le i \le n \\ 0 & \text{o.w.} \end{cases}$$

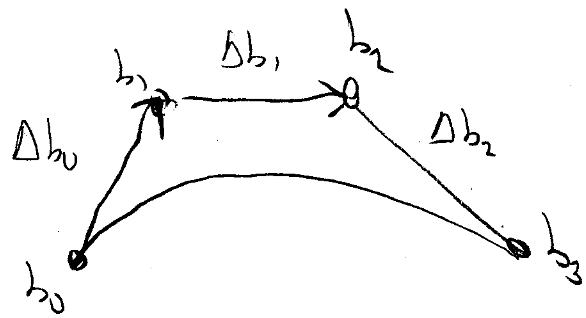
$n = 2$

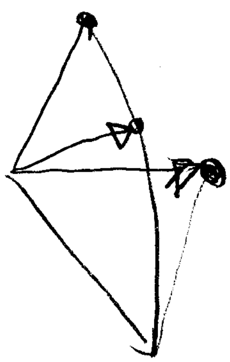
$B_0^2 = \binom{2}{0}(1-t)^2 \qquad B_1^2 = \binom{2}{1} t(1-t) \qquad B_2^2 = \binom{2}{2} t^2$

$= 1 - 2t + t^2 \qquad 2t - 2t^2 \qquad t^2$

$\sum B_i^2 = 1$

Derivative of a Bezier Curve





vector curve $(\Delta b_0, \Delta b_1, \Delta b_2)$

$\Delta b(t)$

$$\frac{\partial b(0)}{\partial t} = d\,\Delta b_0 \qquad d = \text{degree}$$