1. Prove that the LineSweep algorithm presented in class for computing the intersections of \( n \) line segments requires \( O((n + |I|) \log n) \) time where \( I \) is the set of intersections.

**Answer:** The algorithm must maintain a binary tree containing the line segments and a priority queue maintaining the events. Handling an event requires a constant number of operations on each data structure. The queue has at most \( 2n + |I| = O(n^2) \) events so the time for insert and delete-min take \( O(\log(n^2)) = O(\log n) \) time. The tree contains at most \( n \) lines so inserting, deleting, and swapping can all be done in \( O(\log n) \) time using any standard balanced binary search tree.

There are \( O(n + |I|) \) events to process, so the total running time is \( O((n + |I|) \log n) \).

2. Recall that Euler’s formula tells us that for embedded planar graphs, \( V - E + F = 2 \), where \( V \), \( E \), and \( F \) are the number of vertices, edges, and faces (including the infinite face) respectively. Use this formula to compute the maximum number of triangles that may appear in any planar triangulation.

**Answer:** The maximum is achieved when every face (including the infinite face) is a triangle. Thus, if we count the edges from the perspective of the faces, we get 3 edge per face. This counts each edge twice so we get that \( E = \frac{3}{2}F \). Applying Euler’s formula, we get that \( 2 = V - E + F = V - \frac{1}{2}F \) and thus \( F = 2V - 4 \).

3. Suppose we have a set of \( n \) points \( P \) in the plane such that exactly \( h \) of them lie on the convex hull. How many triangles are in any triangulation of \( P \)? Give your answer in terms of \( n \) and \( h \).

**Answer:** As in the previous problem, this is a simple application of Euler’s formula. If we add a point at infinity and triangulate the infinite face, we have to add one triangle per convex hull edge. Thus, by the preceding problem, the number of triangles in the augmented set is \( 2(V + 1) - 4 \). To get our answer, we just subtract off the \( h \) extra triangles added to get \( F = 2V - h - 2 \).
4. Let $P$ be a set of points in the plane. Let $x$ be a point in $CC(P)$, and let $(p_0, \ldots, p_h)$ be a counterclockwise ordering of the vertices of $CH(P)$. If we define $p_{h+1}$ to be $p_0$, then the following formula gives the area of $CC(P)$.

\[
\text{Area}(CC(P)) = \frac{1}{2} \sum_{i=0}^{h} \det \begin{bmatrix} p_i - x \\ p_{i+1} - x \end{bmatrix}
\]

- Why is this formula correct?
  **Answer:** We can split $CC(P)$ into ccw-oriented triangles of the form $\triangle(xp_ip_{i+1})$. Since these triangles are ccw-oriented, their area is the half determinant in the sum. The area of $CC(P)$ is just the sum of the areas of these triangles, which is exactly what the formula computes.

- What happens if the $p_i$’s are given in clockwise order?
  **Answer:** If the $p_i$’s are given in cw order, then the determinants will all be negative and the answer will be $-\text{Area}(CC(P))$.

- Is it still correct if $x \notin CC(P)$?
  **Answer:** Yes.

- Why or why not?
  **Answer:** Intuitively, the triangles $\triangle(xp_ip_{i+1})$ can have either positive or negative determinants. The positive triangles cover an area larger than $CC(P)$. The negative triangles exactly cover the excess. To prove that this is the case, you could simply expand the sum in the formula, observe that it telescopes and write it independent of $x$ as follows.

\[
\frac{1}{2} \sum_{i=0}^{h} \det \begin{bmatrix} p_i - x \\ p_{i+1} - x \end{bmatrix} = \frac{1}{2} \sum_{i=0}^{h} (p_i,x - x)(p_{i+1},y - y) - (p_{i+1},x - x)(p_i,y - y)
\]

(1)

\[
= \frac{1}{2} \sum_{i=0}^{h} p_i,x p_{i+1},y - x_p_i,y p_{i+1},y - x_p_i,x - p_{i+1},x p_i,y + x_p_i,y + x_p_{i+1},x
\]

(2)

\[
= \frac{1}{2} \sum_{i=0}^{h} p_i,x p_{i+1},y - p_{i+1},x p_i,y.
\]

(3)

5. In class we claimed that the intersection of a plane and a paraboloid in $\mathbb{R}^3$ projects to a circle in the plane. Prove it. If you can come up with a nice proof in $\mathbb{R}^3$ go for it. If you are having trouble, prove it for the general case of a hyperplane and a hyperparaboloid using the following steps. Let $p_1, \ldots, p_{d+1}$ be $d+1$ affinely independent points in $\mathbb{R}^d$ and let $p_{i,j}$ denote the $j$th coordinate of the $i$th point. Let $p_i^+$ denote the lifting of point $p_i$ onto the paraboloid, i.e. $p_i^+ = (p_{i,1}, \ldots, p_{i,d}, \sum_{j=1}^{d} p_{i,j}^2)$. 

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• Write a formula to describe the points of $\text{aff}(p_1^+, \ldots, p_{d+1}^+)$ in terms of a set of affine coefficients $\alpha_1, \ldots, \alpha_{d+1}$.

**Answer:** The set of points $q \in \text{aff}(p_1^+, \ldots, p_{d+1}^+)$ are those for which $q_{d+1} = \sum_{i=1}^{d+1} \alpha_i |p_i|^2$ for some $\alpha_i$’s such that $\sum_{i=1}^{d+1} \alpha_i = 1$.

• Prove that the intersection of $\text{aff}(p_1^+, \ldots, p_{d+1}^+)$ projects to the circumsphere of $p_1, \ldots, p_{d+1}$ for the special case when the center of this sphere is the origin.

**Answer:** If the center of the circumsphere is the origin, then $|p_i|^2 = r^2$ for each $p_i$. So, if $q^+$ is both in the plane and the paraboloid then $q_{d+1}^+ = \sum_{i=1}^{d+1} \alpha_i |p_i|^2 = r^2$ and $q_{d+1}^+ = |q|^2$. Thus, $|q| = r$ and so $q$ is on the circle of radius $r$ centered at the origin.

• Prove that the location of the origin does not matter by showing that moving all the points by the same amount along one coordinate axis, does not affect the projected shape.

**Answer:** Let $H = \text{aff}(p_1^+, \ldots, p_{d+1}^+)$, the plane through $p_1^+, \ldots, p_{d+1}^+$. Let $\Pi$ be the paraboloid. The algebra is just as easy if we move the origin to some new point $s$ (rather than just moving one coordinate), resulting in a new plane $H'$ and a new paraboloid $\Pi'$. For any point $q \in H \cap \Pi$, we want to show that $q + s \in H \cap \Pi$.

It will suffice to show that

$$|q + s|^2 = \sum_{i=1}^{d+1} \alpha_i |p_i| |s|^2$$

(where the $\alpha_i$’s are the coefficients for writing $q$ as an affine combination of the $p_i$’s). The LHS corresponds to $\Pi'$ and the RHS corresponds to $H'$. We know that $|q|^2 = \sum_{i=1}^{d+1} \alpha_i |p_i|^2$ because $q \in H \cap \Pi$.

Now,

$$|q + s|^2 = (q + s)^T (q + s) = |q|^2 + s^T s + 2 s^T q.$$ 

Similarly,

$$\sum_{i=1}^{d+1} \alpha_i |p_i + s|^2 = \sum_{i=1}^{d+1} \alpha_i (|p_i|^2 + 2 s^T p_i + s^T s)$$

$$= \sum_{i=1}^{d+1} \alpha_i |p_i|^2 + \sum_{i=1}^{d+1} \alpha_i (2 s^T p_i + s^T s)$$

$$= |q|^2 + s^T s + 2 s^T (\sum_{i=1}^{d+1} \alpha_i p_i)$$

$$= |q|^2 + s^T s + 2 s^T q.$$ 

This completes the proof.