1. Let \( P = \{p_1, \ldots, p_n\} \). Suppose we define

\[
Q = \left\{ x \in \mathbb{R}^d \mid x = \sum_{i=1}^{n} \alpha_ip_i, \sum_{i=1}^{n} \alpha_i = 1, \alpha_i > 0 \right\}.
\]

[Note that the last inequality is strict] Write \( Q \) in terms of \( CH(P) \) and \( CC(P) \).

**Answer:** \( Q = CC(P) \setminus CH(P) \).

I didn’t ask for a proof but if I did, something like the following would have been fine.

First we prove that \( Q \subseteq CC(P) \setminus CH(P) \). If \( x \in Q \) then \( x \in CC(P) \) by the definition of \( CC(P) \). Suppose \( x \in CH(P) \). Let \( S \) be the vertices in a facet of the convex hull containing \( x \). If we write \( x \) as a convex combination of the points in \( P \) then any points in \( P \setminus S \) have coefficient 0 because they are all strictly on one side of \( \text{aff}(S) \).

Now, we prove that the converse, that \( CC(P) \setminus CH(P) \subseteq Q \). For each \( p_i \in P \), consider an infinite ray from \( p_i \rightarrow x \). The ray intersects \( CH(P) \) at some facet with vertices \( S \). We can therefore write \( x \) as a convex combination of the points in \( \{p_i\} \cup S \) as \( x = \sum_{j=1}^{n} \alpha_{ij}p_j \), where \( \alpha_{ij} = 0 \) for any \( j \) such that \( p_j \nsubseteq S \cup \{p_i\} \). Note also that \( \alpha_{ii} > 0 \) because \( x \notin CH(P) \).

Let \( \alpha_i = \frac{1}{n} \sum_{j=1}^{n} \alpha_{ji} \). It is not hard to check that this is indeed an affine combination and that in fact each of the \( \alpha_i \)'s are strictly positive and therefore \( x \in Q \).

2. The centroid of a point set \( P \subset R^d \) is a new point in \( R^d \) whose \( i \)th coordinate is the average of the \( i \)th coordinates of the points of \( P \). Prove that the centroid of \( P \) is contained in \( CC(P) \).

**Answer:** We can write the centroid as \( c = \sum_{i=1}^{n} \frac{1}{n}p_i \). This combination of the \( p_i \)'s is both **affine** and **non-negative** and therefore it is a convex combination.
3. Let us define a new predicate in $\mathbb{R}^2$ as follows.

$$\phi(x, a, b, c) = \frac{\text{sign det} \begin{vmatrix} x & a & b & c \\
|x| & |a| & |b| & |c| \end{vmatrix}}{\text{sign det} \begin{vmatrix} a & b & c \\
1 & 1 & 1 \end{vmatrix}}$$

What does this predicate compute about $x$ with respect to $a$, $b$, and $c$? What is the significance of the denominator? What might you name this predicate? [Hint: This is just a fancy way of describing all of 8th grade math.]

**Answer:** The numerator lifts the points to the cone $z = +\sqrt{x^2 + y^2}$ and then performs a planeside test. The intersection of this cone and plane is a conic section focused at 0. The denominator normalizes the result so that the answer is invariant to the order of the inputs $a$, $b$, and $c$. A good name might be $\text{INCONIC}$.

4. Using the $\text{ccw}$ predicate from class, write a function that tests if a point $x$ is inside a triangle, $\triangle ABC$.

**Answer:** $x \in \triangle ABC \Leftrightarrow \text{ccw}(A, B, x) = \text{ccw}(B, C, x) = \text{ccw}(C, A, x)$.

5. In class, we had to delay the proof of correctness of the Graham Scan algorithm until we had seen how $\text{ccw}$ works. Now we are ready. Prove the correctness of the Graham scan algorithm by proving the following facts.

- **Prove that the output stack contains all of the vertices of the convex hull.**
  
  **Answer:** Every input point gets pushed to the output so it suffices to check that no vertex of $CH(P)$ gets popped from the output. Suppose for contradiction that $v_j \in CH(P)$ and $v_j$ was popped from the output stack. This happens only if there are points $v_i, v_k$ such that $i < j < k$ and $\text{ccw}(v_i, v_j, v_k) = -1$, or equivalently if $\text{ccw}(v_i, v_k, v_j) = 1$. By the initial sorting, $\text{ccw}(v_0, v_i, v_j) = 1$ and $\text{ccw}(v_k, v_0, v_j) = 1$. So, using the method from the previous question, we have that $v_j \in \triangle v_0 v_i v_k$ and therefore, $x \notin CH(P)$.

- **Prove that the output stack contains only the vertices of the convex hull.**
  
  **Answer:** The proof is by induction. Assume that after $k - 1$ points have been considered, the output stack contains $CH(v_0, \ldots, v_{k-1})$. As a base case, the hypothesis holds trivially for a single triangle. Suppose for contradiction that after $k + 1$ points have been considered, the output stack contains some point not on $CH(v_0, \ldots, v_k)$. Since the output ordering is sorted, the edges cannot cross and therefore if the output points are not in convex position, there must be
a sequence of three consecutive points that form a clockwise turn. By our inductive hypothesis, every three consecutive points on the output stack formed a ccw turn before adding $v_k$. The only “new” set of 3 consecutive points on the output stack after adding $v_k$ is the top three points so these must form a cw turn. However, this is exactly the loop condition on the inner loop and therefore the algorithm would have popped the output before adding $v_k$.

- Prove that the vertices are in cyclic order.

**Answer:** To be in cyclic order means that there is a point $x$ in $CC(P)$ such that for every oriented edge $ab$, we have $ccw(a, b, x)$. It suffices to observe that $ccw(a, b, x) \geq 0$. We may set $x = v_0$ and observe that this holds for any $v_i v_j$ with $i < j$ for the initial ordering. Because the ordering is maintained, throughout the algorithm, it holds at the end as well.

**Extra Credit** Recall that a regular $n$-gon can be constructed with a ruler and compass if and only if all of the odd prime factors of $n$ are distinct and of the form $2^{2^k} + 1$ for any $k \in \mathbb{Z}$. Prove that this statement is still true if instead all of the odd prime factors are distinct and of the form $2^k + 1$ for any $k \in \mathbb{Z}$.

**Answer:** It will suffice to prove that if $2^k + 1$ is prime then $k$ is a power of 2. Suppose for contradiction that $k = ab$ for integers $a$ and $b$ with $b$ odd. We see that $2^a + 1$ divides $2^k + 1$ with a little modular arithmetic as follows.

$$2^k + 1 \equiv (2^a)^b + 1 \equiv (-1)^b + 1 \equiv -1 + 1 \equiv 0 \pmod{2^a + 1}.$$