Computational Geometry: Lecture 8

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1 The Point Location Cost of Random Incremental Delaunay Triangulation

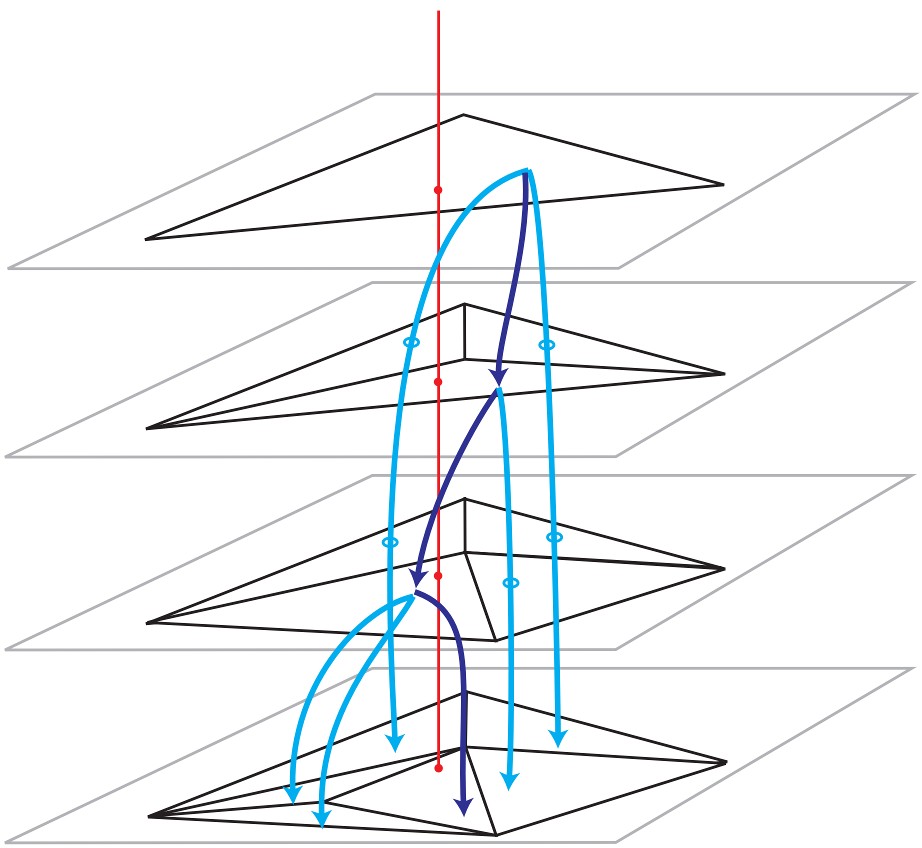

Figure 1: The query is shown in red and the search path through the DAG is shown in dark blue.

The point location for this algorithm works by keeping around all of the triangles seen throughout the course of the algorithm in a data structure called the history DAG. Let $P = \{p_1, \ldots, p_n\}$ be the input points in a random order and let p_i be a particular point whose point location cost we would like to compute. Let D_j be the Delaunay triangulation of $p_1, \ldots, p_j$ and let σ_j be the triangle containing p_i in D_j. Let K_{ij} be the degree of σ_{j-1} in the history DAG. The value of K_{ij} is 0 if and only if $\sigma_{j-1} = \sigma_j$. This means that $K_{ij} \neq 0$ if and only if p_j is a vertex of σ_j. So, the probability can be written as

$$\Pr[K_{ij} \neq 0] = \Pr[p_j \in \sigma_j] = \frac{3}{j}. \tag{1}$$

This step uses that fact that among the first j vertices, all are equally likely to be last.

If K_{ij} is nonzero then its value is at most the degree of p_j in D_j. So, the conditional expected value is

$$E[K_{ij} \mid K_{ij} \neq 0] \leq E[\deg p_j \text{ in } D_j] < 6. \tag{2}$$

We can use the two preceding equations to bound the expected value of K_{ij}.

$$E[K_{ij}] = \Pr[K_{ij} \neq 0] E[K_{ij} \mid K_{ij} \neq 0] < \frac{18}{j}. \tag{3}$$

The total number of triangle checks needed to compute the entire Delaunay triangulation is

$$\sum_{i=1}^{n} \sum_{j=1}^{i} K_{ij}, \tag{4}$$

and its expected value is

$$\begin{aligned} E\left[\sum_{i=1}^{n} \sum_{j=1}^{i} K_{ij}\right] &= \sum_{i=1}^{n} \sum_{j=1}^{i} E[K_{ij}] && (5) \\ &< \sum_{i=1}^{n} \sum_{j=1}^{i} \frac{18}{j} && (6) \\ &= \sum_{i=1}^{n} O(\log i) && (7) \\ &= O(n \log n). && (8) \end{aligned}$$

2 Voronoi Diagrams

As always, let $P \subset \mathbb{R}^2$ be a set of n points in general position.

Definition 2.1. *The* Voronoi cell *of $p \in P$, denoted* $\text{Vor}(p)$*, is the set of points in $\mathbb{R}^2$ that are at least as close to p as to any other $q \in P$. That is,*

$$\text{Vor}(p) = \{x \in R^2 : |x - p| \leq |x - q|, \forall q \in P\}.$$

Definition 2.2. *The* Voronoi diagram *of P, denoted* $\text{Vor}(P)$ *is the complex formed by the collection of Voronoi cells of points of P. The cells of the complex are associated with subsets of $U \subseteq P$ and are the common intersection of the Voronoi cells of each $u \in U$.*

The Voronoi diagram depicts a solution to the so-called Post Office Problem. In this problem, there are a set of post offices P (points in the plane), and we want to know for a given address x (another point in the plane), what is the closest post office to x. We construct the Voronoi diagram of P. If $x \in \text{Vor}(p)$ then p is the closest post office.

I have also heard this problem described in a different context. Let P be the addresses of a collection of muggers. Now muggers tend to mug people close to where they live. Say you are at a street corner x and you just got mugged. If you had a Voronoi diagram of P, you could find the Voronoi cell containing x and there is a good chance that would tell you who mugged you. Clearly this second example is not really an exact science and the problem of actually constructing $\text{Vor}(P)$ involves asking muggers where they live, a practice that is unanimously discouraged by public safety professionals.

Theorem 2.1. *The Voronoi diagram,* $\text{Vor}(P)$ *is dual to the Delaunay triangulation,* $\text{Del}(P)$.

Proof. In order to prove this duality relationship, we need to show the following three bijections.

$$\text{Voronoi cells} \leftrightarrow P \text{ (Delaunay vertices)} \tag{9}$$

$$\text{Voronoi edges} \leftrightarrow \text{Delaunay edges} \tag{10}$$

$$\text{Voronoi vertices} \leftrightarrow \text{Delaunay triangles} \tag{11}$$

The first bijection, Voronoi cells $\leftrightarrow P$, follows straight from the definition of a Voronoi cell.

For the bijection between the edge sets, we will associate a Delaunay edge (p, q) with the Voronoi edge $\text{Vor}(p) \cap \text{Vor}(q)$. We need to show that such a Voronoi edge exists if and only if the corresponding Delaunay edge exists. The Delaunay edge (p, q) exists if and only if there is a circle C through p and p that contains no other points of P in its interior (This fact requires the assumption that the points are in general position). Let x be the center of C. The distances $|x - p|$ and $|x - q|$ are equal to the radius of C. For any point $r \in P$, $|x - r| \leq |x - p| = |x - q|$ because C is empty. Therefore, $x \in \text{Vor}(p) \cap \text{Vor}(q)$ and therefore the Voronoi edge corresponding to (p, q) exists. Conversely, if the Voronoi edge corresponding to (p, q) exists, then any point $x \in \text{Vor}(p) \cap \text{Vor}(q)$ is the center of an empty circle through (p, q), and thus the Delaunay edge (p, q) exists.

Similarly to the case of edges, we will associate the Delaunay triangle $\triangle(pqr)$ with the Voronoi vertex $\text{Vor}(p) \cap \text{Vor}(q) \cap \text{Vor}(r)$. Again, we need to show that this intersection is non-empty if and only if $\triangle(pqr) \in \text{Del}(P)$. We observe that the vertices of the Voronoi Diagram are common intersection of 3 voronoi cells and therefore are equidistant from 3 points of P. That is, the vertices of $\text{Vor}(P)$ are centers of circles that contain 3 points of P. Such a circle cannot contain any other points of P in its interior for other wise, that point would be strictly closer to the center than the points on the circle. This would contradict the assumption that the center is in the Voronoi cells of the points on the circle. So, the three vertices on the circle form a Delaunay triangle. Conversely, every Delaunay triangle has an empty circumcircle whose center is the intersection of the Voronoi cells of its vertices.

□

3 Higher dimensions

It should be clear that there wasn't anything particularly special about the plane when we defined the Voronoi diagram. In fact, all we relied on was distances. So, as you might have guessed, the definition of Voronoi diagrams can be extended to any dimension.

We need to have a new definition to understand what kind of object the Voronoi diagram is in higher dimensions. For this we will talk about a standard object of **combinatorial topology**, known as a *cell complex*.

Definition 3.1. *A* cell complex *is a collection of cells k such that*

1. *if $c_1 \in K$ then all of the faces of c_1 are in K, and*
2. *if $c1, c_2 \in K$ then $c1 \cap c_2$ is a face common to both.*

This definition is not really satisfactory because we haven't yet defined cells or faces. Rather than giving the definition in full generality, we will just consider it in the special case of *polytopal complexes*. This is all we need for this class.

As we saw when we talked about convex hulls, a (convex) polytope is the intersection of a finite set of half-spaces. If the polytopes are bounded, then they can also be represented as the convex closure of a finite point set. Every polytope P has a dimension that is just the dimension of $\text{aff}(P)$. The faces of a polytope are polytopes of lower dimension on the boundary.

Definition 3.2. *A* polytopal complex *is a cell complex whose cells are polytopes.*

Voronoi cells are polytopes. Each cell is the intersection of the halfspaces bounded by the perpendicular bisectors between a point and its Delaunay neighbors. The Voronoi diagram is polytopal complex that covers the ambient space $\mathbb{R}^d$. It is not hard to check that all of the conditions of the definition are satisfied. You are encouraged to do so.