Computational Geometry: Lecture 6

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1 Delaunay Triangulation as Convex Hull

We start with a handy definition.

Definition 1.1. *A (hyper)plane H is a* support (hyper)plane *for a convex set S if H contains at least one point of S and S is contained in a closed halfspace bounded by H.*

It is a basic fact of convex geometry that a point is on the convex hull of S if and only if there exists a support hyperplane.

Recall that the Delaunay triangulation $\mathrm{Del}(P)$ is defined to have the property that every circumcircle is empty of points in P. Recall also that we can check if a point is inside a circumcircle by doing a hyperplane test in $\mathbb{R}^3$ after lifting the points onto the paraboloid. Combining these two ideas we get the following cool fact.

Lemma 1.1. $\mathrm{Del}(P) \cong LH(P^+)$*, where LH denotes the lower convex hull, P^+ is the lifting of P onto the paraboloid, and $\cong$ denotes isomorphism (like in graph theory but with triangles instead of edges).*

Proof. The plane through any set of three lifted points is a support plane with $CC(P^+)$ above it if and only if those points form a Delaunay triangle (before the lifting). □

Lemma 1.2. *For a point set P in general position,* $\mathrm{Del}(P)$ *always exists.*

Proof. This follows from the previous Lemma and the fact that $CH(P^+)$ always exists. □

Lemma 1.3. *The* FlipToDelaunay *algorithm terminates.*

Proof. Let $a(T)$ be the lexicographically sorted list of all the angles in T. We saw last time that flipping an edge that is not locally Delaunay increases the min angle locally. So if triangulation T' is formed from T by flipping one edge that is not locally Delaunay, $a(T') <^{lex} a(T)$. There are only a finite number of triangulations so the process must terminate. □

Lemma 1.4. *If an edge e is not locally Delaunay, then e can be flipped.*

Proof. We will prove the contrapositive: if e cannot be flipped then e is LD. Let a, b, c, d be the four vertices of the two triangles adjacent to e. By assumption e cannot be flipped so a, b, c, d are not in convex position. Without loss of generality, $d \in \triangle abc$. There exists exactly one triangulation of such a point set and it contains all possible edges. Since $\mathrm{Del}(\{a, b, c, d\})$ exists, it must be this triangulation. So, every possible edge, in particular e, is locally Delaunay. □

Theorem 1.1. *The output of the* FlipToDelaunay *algorithm is* $\mathrm{Del}(P)$.

Proof. By the previous Lemma, the algorithm will not terminate while any edges are not LD. It follows that at termination, every edges is LD. Last time we saw that $T = \mathrm{Del}(P)$ if and only if every edge of T is LD. So, the output triangulation is $\mathrm{Del}(P)$. □

2 Small Angles

Theorem 2.1. *Given $P \subset \mathbb{R}^2$ in general position,* $\mathrm{Del}(P)$ *achieves the max-min angle among all triangulations of P.*

Proof. Suppose some triangulation T has a larger min angle than $\mathrm{Del}(P)$, and thus $a(T) < a(\mathrm{Del}(P))$. If T is not Delaunay, we can use the flip algorithm to produce a sequence of triangulations $T = T_1, T_2, \ldots, T_k = \mathrm{Del}(P)$ such that $a(T_{i+1}) < a(T_i)$. It follows that $a(\mathrm{Del}(P)) < a(T)$, a contradiction. □

3 Analysis of the FlipToDelaunay Algorithm

Our proof that the FlipToDelaunay algorithm terminates could be used to give an explicit runtime bound but it wouldn't be very good. We can do much better by using the following Lemma.

Lemma 3.1. *If an edge $\overline{ab}$ is removed by a flip in the* FlipToDelaunay *algorithm at most one time.*

Proof. Let $h_T : CC(P) \to \mathbb{R}$ be the piecewise linear function defined by lifting the vertices of the triangulation T onto the paraboloid.

A Delaunay flip $T \to T'$ switches between the upper and lower hulls of a lifted tetrahedron. So, $h_T > h_{T'}$ and, in fact, the h functions are monotone decreasing as the algorithm proceeds.

Say the edge $\overline{ab}$ is flipped out in favor of an edge $\overline{cd}$. These edges necessarily intersect at some point, call it x. So $h_T(x) > h_{T'}(x)$. Suppose for contradiction that at some later triangulation T'', the edge $\overline{ab}$ is flipped back in, then $h_{T''}(x) = h_T(x) > h_{T'}(x)$, a contradiction. □

Theorem 3.1. *The running time of* FlipToDelaunay *is $O(n^2)$.*

Proof. This follows from the observation that there are only $O(n^2)$ possible edges and each could be removed in at most one flip (by the preceding Lemma). □

4 Higher dimensions

Up until now, we have talked about triangulations as a bag of triangles. Now we'll tighten up our definitions so we can clearly see how to extend them to higher dimensions.

Definition 4.1. *A* k-simplex *is the convex closure of a set of* $k + 1$ *affinely independent points. the points are called the* vertices *and* k *is the* dimension *of the simplex.*

For example, a point is a 0-simplex, an edge is a 1-simplex, a triangle is a 2-simplex, and a tetrahedron is a 3-simplex. We will identify a simplex with its vertex set.

Definition 4.2. *A* face *of a simplex is the convex hull of a subset of the vertices.*

Definition 4.3. *A* simplicial complex *is a family of simplices* K *such that*

1. *if* $\sigma \in K$ *and* σ' *is a face of* σ *then* $\sigma' \in K$*, and*
2. *if* $\sigma \cap \sigma'$ *then they intersect at a common face.*