Computational Geometry: Lecture 10

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1 Geometry, Topology, and Combinatorics

Today we're going to talk about planar graphs. There are many aspects to planar graphs and many perspectives from which to study them. Though we want to focus on the geometry, we will quickly see that we need to depend also on the perspective of topology and of combinatorics.

The combinatorial characterization of planar graphs is probably what you are most familiar with. In that language, we have Kuratowski's Theorem which says that a graph is planar if and only if it lacks a K_5 or a $K_{3,3}$ minor.

The topological characterization is the most immediate intuitively, but not so commonly seen in full rigor by undergraduates. We will not really rectify that situation and will make ample appeal to your intuition about the topology of the plane. For example, we'll assume facts such as the Jordan Curve Theorem and we'll assume definitions such as connectivity of a space.

We saw last time that we can represent a drawing of a planar graph in the plane considering each edge to have two sides and two directions. That is, we took the combinatorial structure of the graph, added some information about faces and the ordering of the edges coming out of a point, and suddenly had topological structure. Where, you might ask, is the geometry?

That will come in the specifics of how we choose to draw or *embed* G in the plane. In particular, we are interested in embeddings that have a nice geometric structure such as when all edges are straight line segments.

2 Planar Graph Basics

The intuitive (topological) definition of a planar graph is just one that we can draw in the plane without any edge crossings. This requires us to be clear about what a *drawing* is, and we'll get to that soon enough.

A purely combinatorial definition comes from a Theorem of Kuratowski.

Theorem 2.1 (Kuratowski's Theorem)**.** *A graph G is planar if and only if it contains neither K_5 nor $K_{3,3}$ as a minor.*

The graphs K_5 and $K_{3,3}$ are respectively the complete graph on 5 vertices and the complete bipartite graph with 3 vertices per part. Recall that a minor of G is a graph derived from G by applying the following two operations in any combination.

- Take a subgraph.
- Contract an edge.

The importance of this characterization is that it doesn't require us to reason directly about the space of all possible drawings of G. In fact, we can use this to prove facts about planar graphs without reference to a drawing at all.

The Jordan Curve Theorem is the one theorem of Topology that we will depend heavily upon throughout our analyses. We will state it here without proof.

Theorem 2.2 (Jordan Curve Thoerem)**.** *Every simple, closed curve in the plane separates the plane into two connected components.*

It says that there is an inside and an outside to every loop and they are not connected. This simple statement is rather intuitive but it's worth noting that its correct proof is often found at the culmination of a full semester course on topology.

3 Embedding Planar Graphs

We are generally comfortable with the notion of a drawing of a graph. We use them all the time, even for non-planar graphs, as a way to get our head around basic principles in graph theory. To speak about drawings of graphs more formally, we will have to introduce some basic topological preliminaries.

Definition 3.1. *An* embedding $\Phi(G)$ *of a graph* $G = (V, E)$ *into the plane is an injective map* $\phi_V : V \to \mathbb{R}^2$ *and a collection of continuous maps* $\psi_e : [0, 1] \to \mathbb{R}^2$ *for each edge* $e \in E$ *such that the following properties hold.*

- *For an edge* e*,*$\psi_e(0)$ *and* $\psi_e(1)$ *map to the endpoints of* e*.*
- *The embeddings of any pair of edges are disjoint or they intersect at a common endpoint.*

Definition 3.2. *The* faces *of an embedding of a planar graph are the connected components of* $\mathbb{R}^2 \setminus \operatorname{im} \Phi(G)$*.*

Recall Euler's Theorem for the plane.

Theorem 3.1. *If* $\Phi(G)$ *is an embedding of a planar graph with* n *vertices,* m *edges, and* f *faces, then the* $n - m + f = 2$*. Moreover,* $f \leq 2n - 4$ *and* $e \leq 3n - 6$ *with equality only for maximally planar graphs (i.e. triangulations).*

Definition 3.3. *A graph* G *is* k-connected *if* $|E| > k$ *and removing* $\leq k$ *vertices will not disconnect it.*

The condition that $|E| > k$ is standard and not really necessary except that it is almost always required. Thus, putting it in the definition avoids us having to add in the extra condition every time we use it. An important equivalent characterization of k-connectivity due to Menger is as follows.

Theorem 3.2 (Menger's Theorem)**.** *A graph $G = (V, E)$ is k-connected if and only if there exist k vertex disjoint paths between every pair of vertices u, v in V.*

You probably saw this one in your graph theory class, and it is not a theorem of geometry so we will skip the proof.

Lemma 3.1. *If a graph planar and 2-connected then every face is a cycle.*

Proof Sketch. Suppose for contradiction that some face F has a vertex v that appears more than once. Then this vertex must separate the graph. There is a simple closed curve C around the shortest tour T in the boundary of F starting and ending at a repeated vertex v. the Jordan Curve theorem implies that C divides the plane into two parts. If T is not the whole boundary of F, then there is some part of in each component of $\mathbb{R}^2 \setminus C$. However, we can draw C so that it only touches G at v and thus every path from the inside of C to the outside goes through v. It follows that v separates G and thus G is not 2-connected, a contradiction. □

Definition 3.4. *A* non-separating cycle *D of a graph G is an induced cycle in G such that $G \setminus C$ is connected.*

If you are ever digging through historical literature, you may also see the term *peripheral polygon* to refer to this same notion. Non-separating cycles are a useful definition because of their intimate connection with the faces of a planar graph.

Theorem 3.3. *If $G = (V, E)$ is a 3-connected planar graph then the faces of G in any embedding in the plane are exactly the non-separating cycles of G.*

Before we jump into the proof, let's reflect for a second why this is a really cool theorem. It gives a characterization of the faces of the graph, that is to say, a topological object, in purely combinatorial terms. It says that we can speak coherently about the faces of a 3-connected planar graph even if we don't have a particular drawing of it.

We have already seen that Euler's formula allows us to talk about the *number* of faces in the graph independent of the embedding.

We have a Theorem, let's check that the hypothesis is necessary. We can construct a simple example of a graph that is not 3-connected, and has no non-separating cycles, but certainly has some faces. Consider for example $K_{2,3}$.

Proof of Theorem 3.3. Let C be a non-separating cycle. By definition, $G \setminus C$ is connected. Suppose for contradiction that C is not a face of G. Then there must be some part of G on both the inside and the outside of C. So, $\Phi(G) \setminus \Phi(C)$ is disconnected. This contradicts the assumption that $G \setminus C$ is connected.

For the converse, let C be a face of $\Phi(G)$. First, we must show that C is an induced cycle in G. Suppose it is not. Then, there is some edge e between non-adjacent vertices of C. This edge splits C into two pieces. Since C is a face, all of the drawing lies on one side of C and so we can use the Jordan Curve Theorem to imply that the ends of e separate the graph. This contradicts the assumption that G was 3-connected.

Now, we will show that $G \setminus C$ is connected. Suppose for contradiction that $G \setminus C$ is disconnected. Then let x and y be two disconnected vertices in $G \setminus C$. By Menger's Theorem, there exist 3 disjoint paths from x to y in G. Because x and y are disconnected in $G \setminus C$, there must be a vertex of each of these paths C. Since C is a face, we can add a vertex at a point inside C and can draw non-crossing edges to the vertices a, b, c. By contracting the edges in all of these paths, we get $K_{3,3}$ as a minor, with partition $(\{a, b, c\}, \{x, y, z\})$. This contradicts the assumption that $\Phi(G)$ is a planar embedding. □

4 Edge-maximal planar graphs

Definition 4.1. *A planar graph G is* edge-maximal *if no edge can be added while preserving planarity. Such a graph is also called a* plane triangulation *and has only triangle faces.*

We won't prove it now, but I will ask you to prove following on your next homework.

Lemma 4.1. *Every plane triangulation with at least 4 vertices is 3-connected.*

This implies that the triangles in a plane triangulation are determined by any embedding.

5 Straight-line embeddings

As long as we are drawing graphs, a natural question comes to mind: given a planar graph G, can we always draw G so that all the edges are straight line segments? The answer is **YES**. We'll in fact see two proof of this Theorem in this class. The first will be today and will proof just this fact. The second will be more elaborate but will in fact prove something much stronger.

Theorem 5.1. *Given a planar graph G, there exists an embedding $\Phi(G)$ in the plane such that the image of every edge is a straight line segment, i.e. $\phi_{(u,v)}(t) = t\phi_V(u) = (1-t)\phi_V(v)$.*

We don't actually have all the tools we will need to prove this Theorem today, but we'll give it a start and see where we get stuck. That should guide us to discover what facts we still need to prove.

Proof. First observe that if the Theorem holds for plane triangulation graphs then it holds for all planar graphs. This is because every planar graph is a subgraph of a plane triangulation (just add edges until you can add any more). If a graph has a straight-line embedding, then the "same" embedding works for any subgraph.

We proceed by induction on the number of vertices $n = |V|$. As a base of the induction, let $n = 3$, then we have a single triangle that we can easily draw with straight edges.

For the inductive step, will remove a vertex and add in edges until we have a plane triangulation again. Euler's formula implies that some vertex v must have degree at most 5. This is the vertex we want to remove.

We will have to add up to 2 edges to get the graph to be edge-maximal again. These edges bound some new triangle faces in the graph and these triangle faces must share a vertex. This follows because the hole left when we remove v has at most 5 sides and all triangulations of a polygon with at most 5 sides have this property. You can easily check all cases.

By induction, the new graph has a straight-line embedding. Now, because triangulations are 3-connected, the Theorem we proved earlier implies that the set of triangles in the embedding are determined. Since the triangles used to fill the hole left by v all share a vertex, the corresponding cycle in the straight-line embedding is star-shaped. Thus, after a slight perturbation, we can add v to the interior of that hole, and draw straight edges to its old neighbors in G. This gives us the desired embedding and completes the proof.

□