

Computational Geometry: Lecture 1*

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1 The Basics

Definition 1.1. Computational Geometry *is the study of algorithms and data structures for*

- 1. constructing geometric objects,*
- 2. computing geometric properties, and*
- 3. answering geometric queries.*

Themes for the course.

1. Computational Geometry extends basic algorithms and data structures to higher dimensions.
2. Linear Algebra is your friend.
3. Randomness is your friend.

*This lecture borrows heavily from a lecture in a similar course in Computational Geometry by Bernard Chazelle at Princeton University.

2 Course Outline

Sorting	$\left\{ \begin{array}{l} \text{Convex Hulls} \\ \text{Delaunay Triangulation} \\ \text{Voronoi Diagrams} \end{array} \right.$
Graphs	$\left\{ \begin{array}{l} \text{Drawing Planar Graphs} \\ \text{Simplicial Complexes} \\ \text{Combinatorial Topology} \end{array} \right.$
Selecting	$\left\{ \begin{array}{l} \text{Geometric Medians} \\ \text{Centerpoints} \\ \text{Statistical Depth} \end{array} \right.$
Search	$\left\{ \begin{array}{l} \text{Geometric Search} \\ \text{Point Location} \\ \text{Nearest Neighbors} \end{array} \right.$
Optimization	$\left\{ \begin{array}{l} \text{Linear Programming} \\ \text{Polytopes} \end{array} \right.$

3 Models of Computation

Why do we need a model of computation? The answer is two-fold. First, we want to *describe* algorithms. Second, we want to *analyze* algorithms. The model of computation tells us what counts as a single operation. There are many choices out there: Turing machine, abacus, cellular automata, pointer machine, etc. In this class, we'll mostly be dealing with the Real-RAM model, which means for us that arithmetic in \mathbb{R} takes $O(1)$ time.

In reality, the computers we deal with cannot store an arbitrary real number. This cause all manner of pain when implementing geometric algorithms from scratch.

4 The Early Computational Geometers

Euclid is, by far, the most influential geometer the world has known. He was, in many respects, a computational geometer in that he dealt extensively with algorithms and geometric constructions. His computational model was the **ruler and compass**. A classic Euclidean construction is the regular pentagon.

We might analyze the “running time” of a ruler and compass construction by the number lines and circles drawn, but it's a little unsatisfying for computer scientists because we get a number like $O(1)$. How do we get the kind of asymptotic analysis that we are so fond of? One idea is to state the problem as follows:

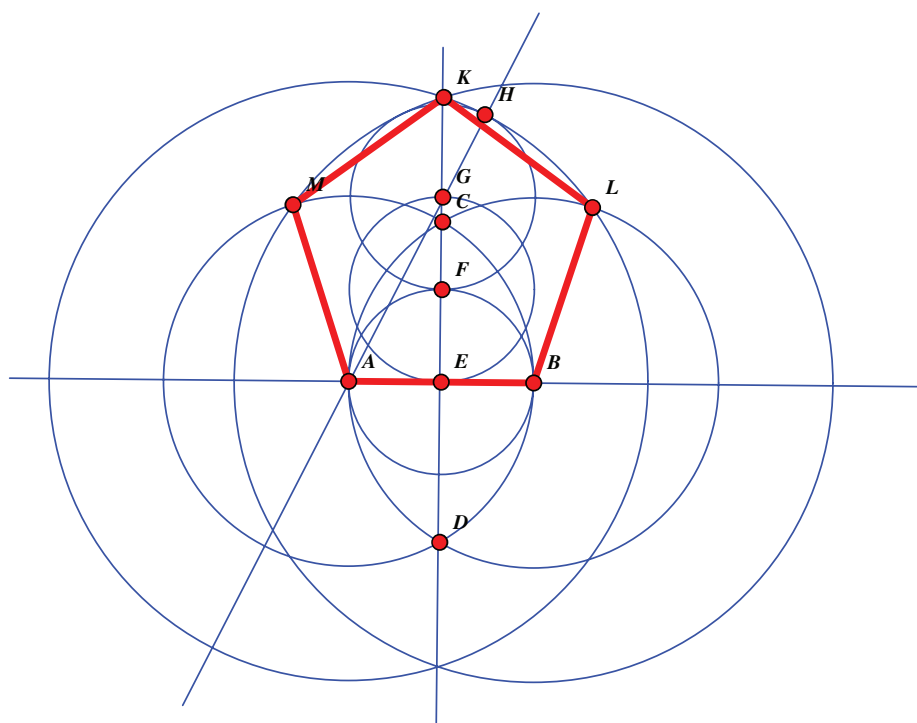


Figure 1: A ruler-and-compass construction of the regular pentagon.

Problem 4.1. *Given a number n , how many steps are required to construct a regular n -gon with ruler and compass?*

This seems like a good, well-formed question, but it assumes that there is a general algorithm for drawing polygons. We've seen the 5-gon construction. Can we generalize it to do a 7-gon. I'm sure that Euclid spent a lot of time looking for a construction of the regular 7-gon. He never found it, nor did anyone else. It took over 2000 years to learn definitively that no such construction could exist. So, it looks like we'll have to revise our problem statement, replacing our algorithms hat with a complexity hat, and answer a decision problem.

Problem 4.2. *Given a number n , does there exist a ruler-and-compass construction of a regular n -gon?*

The first major progress on this problem came from Gauss, who gave a sufficient condition in the 18th century.

Theorem 4.1 (Gauss, 1796). *If the odd prime factors of n are distinct Fermat primes (of the form $2^{2^k} + 1$ for some $k \in \mathbb{Z}$), then there exists a ruler-and-compass construction*

The first several Fermat primes are 3, 5, 17, 257, and 65537. Erchinger was able to use the proof of this Theorem to give an explicit construction of the 17-gon just a few years later. Gauss only gave a sufficient condition for the existence of a construction, but he conjectured that it was also necessary. This was settled 40 years later by Pierre Wantzal.

Theorem 4.2 (Wantzal, 1837). *There exists a ruler-and-compass construction of the n -gon if and only if all of the odd prime factors of n are distinct Fermat primes.*

In 1832, even before Wantzal proved the necessary condition, Friederich Julius Richelot gave a construction for the 257-gon. This extreme act of mathematical patience and diligence is eclipsed by that of Johann Gustav Hermes, who, in 1896, presented a construction for the 65537-gon¹.

4.1 Coordinates

The ruler and compass as a model of computation makes no reference to coordinates. In some ways, this is a feature and not a bug. All of the constructions are independent of scale or translation. A coordinate system was seen as an artificial construction that is not intrinsic to the geometric objects under study. This approach to geometry is known as *Synthetic Geometry* and it is the primary approach to geometry as presented in many high school curricula.

The use of coordinates for geometry was introduced by Descartes in the 17th century. The Cartesian grid allowed for the proof of geometric facts by way of algebraic manipulation of the coordinates. This is known as *Analytical*

¹The construction is almost certainly wrong but who's going to check?

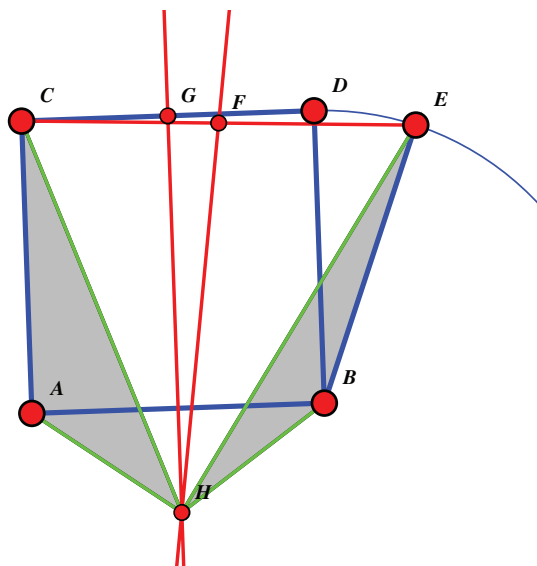


Figure 2: An incorrect proof by picture.

Geometry. It is much closer to the model we will be using to represent points in the algorithms in this class.

The debate between Analytical and Synthetic Geometry raged in the 19th century. One interesting hybrid was proposed by a student of Gauss named Möbius (yes, *that* Möbius). He introduced the use of *Homogenous Coordinates* to the field of *Projective Geometry*. Without going into the details of what this means, I will mention that homogenous coordinates will be important in this class because they allow us to do all kinds of computation as linear algebra (remember: Linear Algebra is your friend).

5 Proof by Picture

When it comes to proving theorems about geometric objects pictures can help, but they can also get you into trouble. We have a lot of visual intuition about space. The key is to use that intuition to guide you towards a correct proof. The intuition is not a proof. Here is an example of a bad proof by picture.

The Construction. The quadrilateral $ABDC$ is a unit square. The segment \overline{EB} also has unit length. The lines through GH and FH are the perpendicular bisectors of \overline{CD} and \overline{CE} respectively.

Theorem 5.1 (Not a real theorem). *Given the above construction, with $\alpha = \angle(DBE) > 0$, the picture (Figure 2) implies that $\alpha = 0$.*

Proof. Letting $|\overline{AB}|$ denote the length of the line segment \overline{AB} , we can derive the following equivalences. First, $|\overline{AH}| = |\overline{BH}|$ because H is on the perpen-

dicular bisector of \overline{AB} . Next, $|\overline{CH}| = |\overline{EH}|$ because H is on the perpendicular bisector of \overline{AB} . We know that $|\overline{AC}| = |\overline{BE}| = 1$ by construction so the triangles $\triangle(ACH)$ and $\triangle(BEH)$ are congruent (Side-Side-Side rule). This implies that the corresponding angles are equal. In particular $\angle CAH = \angle EBH$. Because $\triangle(ABH)$ is isosceles, $\angle HAB = \angle HBA$. However, $\angle CAH = \pi/2 + \angle HAB$, $\angle EBH = \pi/2 + \angle HBA\alpha$, and these two angles are equal. By simple cancellation, we derive that $\alpha = 0$. \square

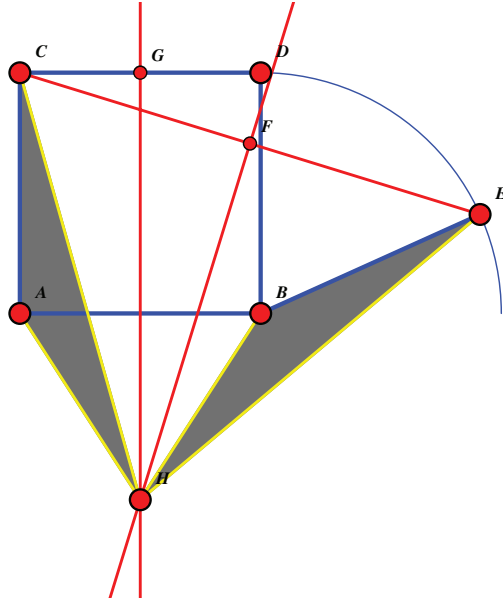


Figure 3: A more accurate picture shows the error in the proof. The triangle on the right had been inverted by incorrectly placing the intersection of the bisectors too high.