The Line Segment Intersection Problem

Input: A set of \( n \) line segments
Output: Report all intersections

Naive: \( O(n^2) \)
Goal: \( O(n \log n + |I|) \) Today \( O((n+|I|) \log n) \)
Worst case: \( |I| = \Omega(n^2) \)

Motivation: Map Overlay

Algorithm: sweep line
Optimized: Incremental
Map overlay prob

Segmen\(t\): \(S = \{s_1, \ldots, s_n\}\) (no vertical seg)

Output: Break all segments into subseg s.t.
Two subseg intersect only at endpoints

Sweep Line Alg

Let \(P = \text{endpoints of segs}\)
\(l = \text{horizontal line disjoint from PVI}\)
\(l\) linearly orders \(S\)
The order only changes at PVI
Store order in Balanced BST

Events = PVI

Idea: sweep \(l\) top to bottom stopping at events.
But we do not know I!

Compute events in $I$ just-in-time.

Claim: If next event is the intersection of $S \& S'$ then $S \& S'$ are neighbors.

Priority Queue $Q_e$ of events

Inductively: $Q_e$ contains

1) $P$

2) Neighbors into (below, $I$).
Handling Events

\[ U(p) = \text{subseg with upper end point } p \]
\[ L(p) = \text{"lower"} \]
\[ C(p) = \text{"intersection } p. \]

Procedure HandleEvent(P, point, T, the Q queue)

0) Use C(p) form new subseg add to U(p) & L(p).
1) Use L(p) delete(s, T)
2) Use U(p) insert(s, T)
3) A new neighbors add intersection to Q.

Let most subseg.

Alg runs in \( O(m \log n) \) Time
there are at most \( m \) delete/inserts into T & Q
How many Subseg?

To show: \( \#\text{Subseg} = O(n + 1) \)

**Embedded Planar Graph** \( (G = (V, E)), \psi: G \to \mathbb{R}^2 \)

- \( \varphi(e) = \text{path} \)
- \( \varphi(e) \cap \varphi(e') = \text{only end points} \)

**Eulers Formula** \( (G, \psi) \)

- \( n_v = \# \text{vertices} \)
- \( n_e = \# \text{edges} \)
- \( n_f = \# \text{connected boundaries (faces)} \)
- \( c = \# \text{connected components} \)

Then \( n_f - n_e + n_v = 2c \)

\[ 5 - 4 + 3 = 2 \cdot 2 = 4 \]

\[ n_v = 5 \quad n_e = 4 \]

\[ n_f = 3 \]

\[ c = 2 \]
Claim \[ 3\eta v \geq \eta e \]

Prove:

1. add edge until \( G \) is connected
2. "each face size in 3."
3. no parallel edges.

\[ 3\eta_f \leq 2\eta e \quad \eta_f \leq \frac{1}{3}\eta e \]

\[ \frac{1}{3} \eta e - \eta e + \eta v \geq 2 \]

\[ -\frac{1}{3} \eta e + \eta v \geq 2 \]

\[ \eta v \geq \eta + \frac{1}{3} \eta e \]

\[ 3\eta v > \eta e \]

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Sweep line is \( O(n + \delta) \log n \) time.

Wrong: We sorted the intersection!